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MATRIX PADÉ APPROXIMATION: RECURSIVE COMPUTATIONS*[1]

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Abstract

In this paper we consider computational aspect of the matrix Padé approximants whose definitions and properties were considered in an accompanying paper. A three-term recursive approach for the computation is established.

In [1] the authors have studied the general matrix Padé approximation problem. It turned out that it is necessary to consider not only left and right approximants, a duality imposed by the non-commutativity of the matrix multiplication, but also type I and type II approximants, depending on the normalization of the denominator. In this paper, we shall consider a recursive method for computing the approximants. We use the definitions and notations from [1].

We assume that $V, W \in Z_+^{p\times 1}$ and $U \in Z_+^{m\times 1}$, so that we do not have to mention this condition every time. On the other hand, we mainly consider the computation of type I right MPAs. The computation of the second type will be shown to be equivalent to the computation of the first type under some conditions. So the right subscript I or II in the notations will be deleted if there is no confusion. Consider the set

$$[V,U,W] = \{(N,M) : NM^{-1} \in [V,U,W]^f\}.$$

The problem we want to solve here is to compute $[V,U,W]$ from two "previous" ones. We shall need some normality condition for f which is different from the one we have defined earlier. We call the new concept I-normality, defined as follows:

Definition 1 (I-normality). *If for any V, U and W which satisfy condition*

$$\sum_{k=1}^{p} w_{kj} = \sum_{k=1}^{p} v_{kj} + \sum_{k=1}^{m} u_{kj}, \quad j = 1, 2, \cdots, m, \tag{1}$$

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the matrix $H(V,U,W)$ is nonsingular, then we say that f is I-*normal.*

For $(N,M) \in [V,U,W]$, we introduce the following notations for the elements of the numerator, denominator and residual:

$$N_{ij}(z) = \sum_{k=0}^{v_i} n_{ij}^{(k)} z^k, \quad M_{ij}(z) = \sum_{k=0}^{u_i} m_{ij}^{(k)} z^k, \quad \text{and} \quad (fM - N)_{ij}(z) = \sum_{k=0}^{\infty} e_{ij}^{(k)} z^k.$$

We introduce the following matrices:

$$N(V,U,W,V')^T = [N_1^{(v_1'+1)}(V,U,W)^T \cdots N_1^{(v_1)}(V,U,W)^T \cdots$$
$$N_p^{(v_p'+1)}(V,U,W)^T \cdots N_p^{(v_p)}(V,U,W)^T],$$
$$M(V,U,W)^T = [M_1^{(0)}(V,U,W)^T \cdots M_1^{(u_1)}(V,U,W)^T$$
$$\cdots M_m^{(0)}(V,U,W)^T \cdots M_m^{(u_m)}(V,U,W)^T],$$

and

$$E(V,U,W,W')^T = [E_1^{(w_1+1)}(V,U,W)^T \cdots E_1^{(w_1')}(V,U,W)^T \cdots E_p^{(w_p+1)}(V,U,W)^T$$
$$\cdots E_p^{(w_p')}(V,U,W)^T],$$

where

$$N_i^{(k)}(V,U,W) = [n_{i1}^{(k)}\ n_{i2}^{(k)}\ \cdots\ n_{im}^{(k)}], \quad M_i^{(k)}(V,U,W) = [m_{i1}^{(k)}\ m_{i2}^{(k)}\ \cdots\ m_{im}^{(k)}],$$
$$E_i^{(k)}(V,U,W) = [e_{i1}^{(k)}\ e_{i2}^{(k)}\ \cdots\ e_{im}^{(k)}].$$

The degrees $V'+1$ and orders W' satisfy $V'+1 \in Z_+^{p\times 1}$ and $V' \le V, W' \ge W$. It can be seen that $N(V,U,W,-1)$ is the coefficient matrix of N, $M(V,U,W)$ is the coefficient matrix of M, and $E(V,U,W,\infty)$ is the coefficient matrix of $R = fM - N$. Hence, from the definition of MPA, we have

$$H(V'+1,U+1,V+1)M(V,U,W) = N(V,U,W,V'), \tag{2}$$
$$H(V+1,U+1,W+1)M(V,U,W) = 0, \tag{3}$$
$$H(W+1,U+1,W'+1)M(V,U,W) = E(V,U,W,W').$$

Lemma 1. *Let f be* I-*normal. Then for any V,U,W satisfying* (1), *we have*

(i) *The matrix*

$$\begin{bmatrix} N(V,U,W,V') \\ E(V,U,W,W') \end{bmatrix} \tag{5}$$

is nonsingular, where

$$-1 \le V' \le V, \quad W' \ge W \text{ and } |V-V'| + |W'-W| = m. \tag{6}$$

(ii) *The leading coefficient matrix of the denominator M*

$$M^{hc}(V,U,W)^T = \left[M_1^{(u_1)}(V,U,W)^T \cdots M_m^{(u_m)}(V,U,W)^T\right]$$

is nonsingular.

(iii) *For V', W' satisfying* (6), *and $W'' \ge W'$ and $(V,\ U+1,\ W'')$ satisfying* (1), *the matrix*

$$\begin{bmatrix} N(V,U,W,V') \\ E(V,U,W,W') \end{bmatrix} - \begin{bmatrix} N(V,U+1,W'',V') \\ 0 \end{bmatrix} \tag{7}$$

is nonsingular.

Proof. (i) For $N \in Z_+^{p\times 1}$, define

$$\theta_N^{(j)} = \left[\theta_{n_j,n_1}\theta_{n_j,n_2} \cdots \theta_{n_j,n_p}\right], \quad I_N^{(j)} = \left[\theta_{n_j,n_1} \cdots \theta_{n_j,n_{j-1}} I_{n_j} \theta_{n_j,n_{j+1}} \cdots \theta_{n_j,n_p}\right],$$

where $\theta_{q,k} \in C^{q\times k}$ is a zero matrix. For $N_i \in Z_+^{p\times 1}, i = 0, 1, \cdots, k$, and $N_0 \le N_1 \le \cdots \le N_k$, let

$$L^{(j)}(N_0, N_1, \cdots, N_k) = \begin{bmatrix} I_{N_1-N_0}^{(j)} & \theta_{N_2-N_1}^{(j)} & \cdots & \theta_{N_k-N_{k-1}}^{(j)} \\ \theta_{N_1-N_0}^{(j)} & I_{N_2-N_1}^{(j)} & \cdots & \theta_{N_k-N_{k-1}}^{(j)} \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{N_1-N_0}^{(j)} & \theta_{N_2-N_1}^{(j)} & \cdots & I_{N_k-N_{k-1}}^{(j)} \end{bmatrix},$$

and

$$L(N_0, N_1, \cdots, N_k)^T = [L^{(1)}(N_0, N_1, \cdots, N_k)^T \; L^{(2)}(N_0, N_1, \cdots, N_k)^T \cdots L^{(p)}(N_0, N_1, \cdots, N_k)^T].$$

Then

$$H(V'+1, U+1, W'+1) = L(V', V, W, W') \begin{bmatrix} H(V'+1, U+1, V+1) \\ H(V+1, U+1, W+1) \\ H(W+1, U+1, W'+1) \end{bmatrix}. \quad (8)$$

Let

$$I(U)^T = \left[\theta_U^T \; I_U^{(1)^T} \cdots \theta_U^T \; I_U^{(m)^T}\right], \quad J(U)^T = \left[I_U^{(1)^T} \; \theta_U^T \cdots I_U^{(m)^T} \; \theta_U^T\right],$$

where $\theta_U \in C^{1\times|U|}$ is a zero vector. Then

$$\begin{aligned} \operatorname{rank} H(V'+1, U+1, W'+1) &= \operatorname{rank}\{H(V'+1, U+1, W'+1)[I(U) \; M(V, U, W)]\} \\ &= \operatorname{rank} \begin{bmatrix} H(V', U, V) & H(V'+1, U+1, V+1)M(V, U, W) \\ H(V, U, W) & 0 \\ H(W, U, W') & H(W+1, U+1, W'+1)M(V, U, W) \end{bmatrix} \\ &= \operatorname{rank} H(V, U, W) + \operatorname{rank} \begin{bmatrix} N(V, U, W, V') \\ E(V, U, W, W') \end{bmatrix}. \end{aligned} \quad (9)$$

Now (5) follows from (9) and the I-normality of f.

(ii) Let

$$\bar{M}(V, U, W)^T = [M_1^{(0)}(V, U, W)^T \cdots M_1^{(u_1-1)}(V, U, W)^T \cdots M_m^{(0)}(V, U, W)^T \cdots M_m^{(u_m-1)}(V, U, W)^T].$$

Then

$$\begin{aligned} &\operatorname{rank} H(V+1, U, W+1) = \operatorname{rank}\{H(V+1, U, W+1)[I(U-1) \; \bar{M}(V, U, W)]\} \\ &= \operatorname{rank}[H(V, U-1, W) \; H(V+1, U, W+1)\bar{M}(V, U, W)] \\ &= \operatorname{rank}\left[H(V, U-1, W) \; -\left[C^1(V, U, W) \cdots C^m(V, U, W)\right] M^{hc}(V, U, W)\right] \\ &= \operatorname{rank}\left\{\left[H(V, U-1, W) \; C^1(V, U, W) \cdots C^m(V, U, W)\right] \begin{bmatrix} I_{|U|-m} & 0 \\ 0 & -M^{hc}(V, U, W) \end{bmatrix}\right\} \end{aligned}$$

$$= \operatorname{rank}\left\{ H(V,U,W)\begin{bmatrix} I_{|U|-m} & 0 \\ 0 & -M^{hc}(V,U,W) \end{bmatrix}\right\}$$

$$= \operatorname{rank}\begin{bmatrix} I_{|U|-m} & 0 \\ 0 & -M^{hc}(V,U,W) \end{bmatrix},$$

where

$$C^j(V,U,W)^T = \left[T^{v_1-u_j+1}_{w_1-v_1,1}(f_{1j})^T \cdots T^{v_p-u_j+1}_{w_p-v_p,1}(f_{pj})^T\right].$$

Hence conclusion (ii) holds.

(iii) Let

$$M_1(V,U,W) = \begin{bmatrix} M_1^{(1)}(V,U,W) - M_1^{(1)}(V,U+1,W'') \\ \vdots \\ M_1^{(u_1)}(V,U,W) - M_1^{(u_1)}(V,U+1,W'') \\ M_1^{(u_1+1)}(V,U+1,W'') \\ \vdots \\ M_m^{(1)}(V,U,W) - M_m^{(1)}(V,U+1,W'') \\ \vdots \\ M_m^{(u_m)}(V,U,W) - M_m^{(u_m)}(V,U+1,W'') \\ M_m^{(u_m+1)}(V,U+1,W'') \end{bmatrix}.$$

Noting that $M(0) = I$ and that the leading coefficient matrix $M^{hc}(V,U+1,W)$ is nonsingular, we have by (8) that

$$\begin{aligned} &\operatorname{rank} H(V',U+1,W') \\ &\quad= \operatorname{rank}\{H(V',U+1,W')[M_1(V,U,W)\ \ J(U)]\} \\ &\quad= \operatorname{rank}\begin{bmatrix} N(V,U,W,V') - N(V,U+1,W'',V') & H(V',U,V)) \\ 0 & H(V,U,W) \\ E(V,U,W,W') & H(W,U,W') \end{bmatrix} \\ &\quad= \operatorname{rank}\left\{\begin{bmatrix} N(V,U,W,V') \\ E(V,U,W,W') \end{bmatrix} - \begin{bmatrix} N(V,U+1,W'',V') \\ 0 \end{bmatrix}\right\} + \operatorname{rank} H(V,U,W). \end{aligned}$$

Thus conclusion (iii) holds.

Corollary 2. Suppose f is I-normal. Then

(i) $E(V,U,W,W')$ is nonsingular, when $W' \geq W$ and $|W' - W| = m$.

(ii) $N(V,U,W,V')$ is nonsingular, when $-1 \leq V' \leq V$ and $|V - V'| = m$.

These are two special cases of conclusion (i) of Lemma 1.

Corollary 3. Suppose f is I-normal. If U satisfies $|u_i - u_j| \leq 1\ \forall i,j,\ u_1 \geq u_2 \geq \cdots \geq u_m$, then the two types of MPA problems are equivalent.

Proof. From conclusion (ii) of Lemma 1, it follows that the leading coefficient matrix M^{hc} of $M(z)$ is nonsingular. Multiply (N,M) from the right with M^{hc} and the leading coefficient matrix of M will become the unit matrix. Then the denominator is made $C1$ canonical. On the other hand, $[V,U,W]_{II}$ exists uniquely, since $H_{II}(V,U,W) = H_I(V+1,U,W+1)$. Then the equivalency is proved.

This is the reason why we do not have consider the computation of MPAs of type II.

Theorem 4. *Suppose $m \ge p$ and f is I-normal. Then, for given V, U, W satisfying (1), and $W' \ge W$, $|W' - W| = m$,*

(i) *There exists a nonsingular matrix $Q(V,U,W,W') \in C^{m\times m}$ such that*

$$[V+1, U+1, W'+1] = z[V,U,W]Q(V,U,W,W') + [V+1,U,W+1], \tag{10}$$

$$Q(V,U,W,W') = -E(V,U,W,W')^{-1}E(V+1,U,W+1,W'+1). \tag{11}$$

(ii) *If $W' \ge W+1$, there exists a rank p matrix $D(V,U,W,W') \in C^{m\times m}$ such that*

$$[V+1,U+1,W'+1] = z[V,U,W]D(V,U,W,W') + [V,U+1,W'], \tag{12}$$

$$D(V,U,W,W') = -E(V,U,W,W')^{-1}H(W+2,U+2,W'+2)M(V,U+1,W'). \tag{13}$$

(iii) *If $W' \ge W+1$, there exists a rank p matrix $T(V,U,W,W') \in C^{m\times m}$ such that*

$$[V-1,U+1,W'-1] = [V,U,W]T(V,U,W,W') + [V,U+1,W'](I - T(V,U,W,W')), \tag{14}$$

$$T(V,U,W,W') = -\begin{bmatrix} N(V,U,W,V-1) - N(V,U+1,W',V-1) \\ E(V,U,W,W'-1) \end{bmatrix}^{-1} \times N(V,U+1,W',V-1). \tag{15}$$

(iv) *If $W' \ge W+1$, there exists a nonsingular matrix $S(V,U,W,W') \in C^{m\times m}$ such that*

$$[V-1,U+1,W'-1] = [V,U,W] + z[V-1,U,W-1]S(V,U,W,W'), \tag{16}$$

$$S(V,U,W,W') = -\begin{bmatrix} N(V-1,U,W-1,V-2) \\ E(V-1,U,W-1,W'-2) \end{bmatrix}^{-1} \begin{bmatrix} N(V,U,W,V-1) \\ E(V,U,W,W'-1) \end{bmatrix}. \tag{17}$$

Proof. (i) Let $(N_1, M_1) = [V,U,W]$, $(N_2, M_2) = [V+1,U,W+1]$, $(N_3, M_3) = [V+1,U+1,W'+1]$. Then by their definitions

$$fM_1 - N_1 = O(z^{W+1}), \quad fM_2 - N_2 = O(z^{W+2}) \quad \text{and} \quad fM_3 - N_3 = O(z^{W'+2}),$$

$Q(V,U,W,W')$ satisfies

$$E(V,U,W,W')Q(V,U,W,W') + E(V+1,U,W+1,W'+1) = 0.$$

From Lemma 1 , (10) and (11) hold and $Q(V,U,W,W')$ is nonsingular.

(ii) Similarly to (i), $D(V,U,W,W')$ satisfies

$$E(V,U,W,W')D(V,U,W'W') + H(W+2,U+2,W'+2)M(V,U+1,W') = 0.$$

From Lemma 1, (12) and (13) hold, and

$$\begin{aligned}\operatorname{rank} D(V,U,W,W') &= \operatorname{rank}\,[H(W+2,U+2,W'+2)M(V,U+1,W')]\\ &= \operatorname{rank}\left\{\begin{bmatrix} H(W+2,U+2,W'+1)\\ H(W'+1,U+2,W'+2)\end{bmatrix} M(V,U+1,W')\right\}\\ &= \operatorname{rank}\begin{bmatrix} 0\\ H(W+2,U+2,W'+1)\end{bmatrix} = p.\end{aligned}$$

(iii) Since $T(V,U,W,W')$ satisfies

$$N(V,U,W,V-1)T(V,U,W,W') + N(V,U+1,W',V-1)(I-T(V,U,W,W')) = 0,$$

$$E(V,U,W,W'-1)T(V,U,W,W') = 0,$$

it follows from Lemma 1 that (14) and (15) hold and that

$$\operatorname{rank} T(V,U,W,W') = \operatorname{rank} N(V,U+1,W',V-1) = p.$$

(iv) Since $S(V,U,W,W')$ satisfies

$$N(V,U,W,V-1) + N(V-1,U,W-1,V-2)S(V,U,W,W') = 0,$$

$$E(V,U,W,W'-1) + E(V-1,U,W-1,W'-2)S(V,U,W,W') = 0,$$

it follows from Lemma 1 that (16) and (17) hold and that $S(V,U,W,W')$ is nonsingular. The proof of the theorem is now completed.

Formulas (10), (12), (14) and (16) can be written in matrix form. These matrices are the coefficient matrices of numerator and denominator. For example, (10) can be written as

$$\begin{aligned}N(V+1,U+1,W'+1,-1) &= \downarrow N(V,U,W',-1)Q(V,U,W,W')\\ &\quad + N(V+1,U,W+1,-1),\\ M(V+1,U+1,W'+1) &= \downarrow M(V,U,W')Q(V,U,W,W')\\ &\quad + M(V+1,U,W+1),\end{aligned}$$

where

$$\begin{aligned}N(V,U,W,V')^T &= [\theta_m^T\ N_1^{(0)}(V,U,W)^T \cdots N_1^{(v_1)}(V,U,W)^T\\ &\quad \cdots \theta_m^T\ N_p^{(0)}(V,U,W)^T \cdots N_p^{(v_p)}(V,U,W)^T],\\ M(V,U,W)^T &= [\theta_m^T\ M_1^{(0)}(V,U,W)^T \cdots M_1^{(u_1)}(V,U,W)^T\\ &\quad \cdots \theta_m^T\ M_m^{(0)}(V,U,W)^T \cdots M_m^{(u_m)}(V,U,W)^T].\end{aligned}$$

For the coefficient matrix of the residual, there is a similar formula. Therefore, formulas (10)–(17) can be used to compute the MPAs recursively, provided two approximants $[V,U,W]$ and $[V+1,U,W+1]$ or $[V,U,W]$ and $[V,U+1,W']$ are known. If all the numerator degrees are the same, i.e., $U \in Z_+$, we can start from $[V,0,V] = (f^{(V)}(z),\ I_m)$ and $[V+1,0,V+1] = (f^{(V+1)}(z),\ I_m)$.

From Theorem 4 we know that, if $p = m$, all the parameters Q, D, T, S are nonsingular.

Now we establish some recursive relations among the parameters Q, D, T, S.

Theorem 5. *Suppose $m \geq p$ and f is* I*-normal. Then, for given V, U, W satisfying* (1), *and W' satisfying $W' \geq W + 1$, $|W' - W| = m$, we have for the matrices appearing in the previous theorem the following relations:*

$$D(V,U,W,W') = Q(V,U,W,W') - S(V+1,U,W+1,W'+1), \tag{18}$$

$$T(V+1,U,W+1,W'+1) = Q^{-1}(V,U,W,W')D(V,U,W,W'). \tag{19}$$

For $W'' \leq W$ and $|W + 1 - W''| = m$,

$$S(V+2,U,W+2,W'+2) = Q(V,U,W,W') - D(V+1,U-1,W'',W+1). \tag{20}$$

For $W'' \leq W$ and $|W + 1 - W''| = m$,

$$D(V+1,U-1,W'',W+1) = T(V+1,U-1,W'',W+1)Q(V,U,W,W'). \tag{21}$$

Proof. (i) In formula (16), changing the parameter (V,U,W,W') to $(V+1,U,W+1,W'+1)$ and subtracting with formula (10), we get

$$[V+1,U+1,W'+1] - [V,U+1,W'] = z[V,U,W][Q(V,U,W,W') - S(V+1,U,W+1,W'+1)].$$

It follows from (12) that the right-hand side of the above equality is just $z[V,U,W]$ $D(V,U,W,W')$. Therefore relation (18) holds.

(ii) Eliminating $z[V,U,W]$ from (10) and (12), we get

$$\begin{aligned}[V,U+1,W'] &= [V+1,U,W+1]Q^{-1}(V,U,W,W')D(V,U,W,W') \\ &+ [V+1,U+1,W'+1][I - Q^{-1}(V,U,W,W')D(V,U,W,W')].\end{aligned}$$

Comparing this with formula (14), we get relation (19).

(iii) Since

$$\sum_{i=1}^{p}(v_i+1) + \sum_{i=1}^{m}(u_i-1) = \sum_{i=1}^{p} w_i + p - m = \sum_{i=1}^{p} w_i'',$$

(1) holds for $(V+1,U-1,W'')$. In formula (14), take the parameter (V,U,W,W') to be $(V+1,U-1,W'',W+1)$. Then the conditions on conclusion (iii) of the theorem hold. (14) now becomes

$$\begin{aligned}[V,U,W] &= [V+1,U-1,W'']T(V+1,U-1,W'',W+1) \\ &+ [V+1,U,W+1](I - T(V+1,U-1,W'',W+1)).\end{aligned}$$

Substituting it into (10), we have

$$\begin{aligned}&[V+1,U+1,W'+1] \\ &= z[V+1,U-1,W'']T(V+1,U-1,W'',W+1)Q(V,U,W,W') \\ &+ z[V+1,U,W+1]\,[I - T(V+1,U-1,W'',W+1)]\,Q(V,U,W,W') \\ &+ [V+1,U,W+1].\end{aligned} \tag{22}$$

In formula (16), changing the parameter (V, U, W, W') to $(V+2, U, W+2, W'+2)$, we have

$$[V+1, U+1, W'+1] = [V+2, U, W+2] + z[V+1, U, W+1)S(V+2, U, W+2, W'+2). \quad (22)$$

Comparing the coefficient of z of the denominator on the right hand side of (22) and (23), we get

$$Q(V, U, W, W') + M^{(1)}(V+1, U, W+1) = M^{(1)}(V+2, U, W+2) + S(V+2, U, W+2, W'+2), \quad (24)$$

where $M^{(1)}(V, U, W)$ denotes the coefficient matrix of z in the denominator. In formula (12) , take the parameter (V, U, W, W') as $(V+1, U-1, W'', W+1)$. Then we have

$$[V+2, U, W+2] = z[V+1, U-1, W'']D(V+1, U-1, W'', W+1) + [V+1, U, W+1].$$

It follows that

$$M^{(1)}(V+2, U, W+2) = D(V+1, U-1, W'', W+1) + M^{(1)}(V+1, U, W+1).$$

Combining it with (24), we get (20).

(iv) Comparing the leading coefficients of the denominator in (22) and (23), we have

$$M^{hc}(V+1, U, W+1)\,[I - T(V+1, U-1, W'', W+1)]\,Q(V, U, W, W') = M^{hc}(V+1, U, W+1)S(V+2, U, W+2, W'+2).$$

Since $M^{hc}(V+1, U, W+1)$ is nonsingular,

$$S(V+2, U, W+2, W'+2) = [I - T(V+1, U-1, W'', W+1)]\,Q(V, U, W, W').$$

Using (20), we get (21). The proof is completed.

If $p = m$ and $V, U, W, W', W'' \in Z_+$, i.e., $V = v$, $U = u$, $W = w$, $W' = w'$, $W'' = w''$, then $w = v + u$, $w' = w + 1$, $w'' = w$. Let

$$U_u^v = S(V+1, U-1, W, W+1), \quad T_u^v = T(V+1, U-1, W, W+1),$$
$$E_u^v = -D(V-1, U-1, W-2, W-1), \quad Q_u^v = -Q(V-1, U-1, W-2, W-1).$$

Then we have the following relations for these matrices:

Corollary 6. *Suppose f is normal. Then with the notation just introduced, we get the following qd-like relations:*

$$E_u^v = Q_u^v + U_u^{v-1}, \qquad v \geq 1, u \geq 1, \quad (25)$$

$$U_u^{v-1} = E_{u-1}^v - Q_u^{v-1}, \qquad v \geq 1, u \geq 1, \quad (26)$$

$$Q_{u+1}^v = (T_u^{v-1})^{-1} E_u^{v+1}, \qquad v \geq 1, u \geq 1, \quad (27)$$

$$T_u^{v-1} = (Q_u^v)^{-1} E_u^v, \qquad v \geq 1, u \geq 1, \quad (28)$$

where

$$E_0^v = 0, \qquad v = 1, 2, \cdots, \quad (29)$$

$$Q_1^v = c_v^{-1} c_{v+1}, \quad v = 0, 1, \cdots, \tag{30}$$

$$Q_u^0 = 0, \qquad u = 2, 3, \cdots. \tag{31}$$

Proof. All the formulas in (25)–(28) are special cases of Theorem 5 except for $v = 1$ or $u = 1$ in (26), because the parameters Q_u^0 and E_0^v are not defined in Theorem 5. Therefore, the only thing to be verified is that the initial parameters given in (29)–(31) are correct indeed. From the definition of Q_1^v, we have

$$Q_1^v = -Q(v-1, 0, v-1, v) = E(v-1, 0, v-1, v)^{-1} E(v, 0, v, v+1) = c_v^{-1} c_{v+1},$$

i.e., (30) holds.

If $u = 1$ and $v \geq 1$, we get from the definition of U_1^{v-1},

$$U_1^{v-1} = S(v, 0, v, v+1) = -N(v-1, 0, v-1, v-2)^{-1} N(v, 0, v, v-1) = -c_{v-1}^{-1} c_v,$$

i.e., $U_1^{v-1} = -Q_1^{v-1}$. Therefore, (29) is correct.

If $v = 1, u \geq 2$, from (12) and (16) we have

$$[1, u-1, u] = -z[0, u-2, u-2] E_{u-1}^1 + [0, u-1, u-1],$$

$$[0, u, u] = [1, u-1, u] + z[0, u-1, u-1] U_u^0.$$

Hence

$$[0, u, u] = -z[0, u-2, u-2] E_{u-1}^1 + [0, u-1, u-1] + z[0, u-1, u-1] U_u^0.$$

Equating the coefficient of z in the denomimator, we have

$$U_u^0 = E_{u-1}^1 + [M^{(1)}(0, u, u) - M^{(1)}(0, u-1, u-1)].$$

Since $M^{(1)}(0, u, u) - M^{(1)}(0, u-1, u-1) = 0$, (31) is correct.

Using the formulas given in the corollary, we can evaluate all the parameters recursively from the initial parameters.

From the previous corollary, we immediately get the following.

Corollary 7. Suppose f is normal. Then

$$E_u^v = Q_u^v + E_{u-1}^v - Q_u^{v-1}, \quad v \geq 1, u \geq 1,$$

$$Q_{u+1}^v = (E_u^v)^{-1} Q_u^v, E_u^{v+1}, \quad v \geq 1, u \geq 1,$$

where

$$E_0^v = 0,\ v = 1, 2, \cdots; \quad Q_1^v = c_v^{-1} c_{v+1},\ v = 0, 1, \cdots; \quad Q_u^0 = 0,\ u = 2, 3, \cdots.$$

This is the analogue of the qd algorithm for the classical Padé approximants.

References

[1] G.L. Xu, A. Bultheel, Matrix Padé Approximants : Definitions and Properties, submitted. See also : G.L. Xu, A. Bultheel, The Problem of Matrix Padé Approximation, Report TW116, Dept. Computer Science, K.U. Leuven, November 1988.