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OPTIMUM MODIFIED SOR (MSOR) METHOD IN A SPECIAL CASE*

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Abstract

In this paper we study the MSOR method with fixed parameters, when applied to a linear system of equations $Ax = b$ (1), where A is consistently ordered and all the eigenvalues of the iteration matrix of the Jacobi method for (1) are purely imaginary. The optimum parameters and the optimum virtual spectral radius of the MSOR method are also obtained by an analysis similar to that of [5, pp. 277-281] for the real case. Finally, a comparison of the optimum MSOR method with the optimum SOR and AOR methods is presented, showing the superiority of the MSOR one.

§1. Introduction

To solve the linear system of equations:

$$Ax = b, \tag{1.1}$$

where $A \in R^{n,n}$, $b \in R^n$ and $\det(A) \neq 0$, we consider the modified successive overrelaxation (MSOR) method with fixed parameters (see e.g. [5, Chapter 8], [2]). We also assume that A has the form

$$A = \begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix} \tag{1.2}$$

where D_1 and D_2 are nonsingular diagonal matrices. If we partition x and b in (1.1) in accordance with the partitioning of A in (1.2), we can write system (1.1) in the form

$$\begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{1.3}$$

The MSOR method is defined by

$$x^{(m+1)} = L_{\omega,\omega'} x^{(m)} + z_{\omega,\omega'}, \quad m = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$L_{\omega,\omega'} = \begin{bmatrix} (1-\omega)I_1 & \omega F \\ \omega'(1-\omega)G & \omega\omega' GF + (1-\omega')I_2 \end{bmatrix} \tag{1.5}$$

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and

$$z_{\omega,\omega'} = \begin{bmatrix} \omega c_1 \\ \omega\omega' G c_1 + \omega' c_2 \end{bmatrix}. \tag{1.6}$$

In (1.5) and (1.6),

$$F = -D_1^{-1}H,\ G = -D_2^{-1}K,\ c_1 = -D_1^{-1}b_1,\ c_2 = -D_2^{-1}b_2, \tag{1.7}$$

I_1 and I_2 are identity matrices of the same sizes as D_1 and D_2 respectively and $\omega, \omega'(\neq 0)$ are the real relaxation factors. If $\omega = \omega'$, the MSOR method reduces to the SOR method and we write

$$L_\omega \equiv L_{\omega,\omega}.$$

In the following we first find necessary and sufficient conditions for strong convergence and then determine the optimum parameters and the optimum virtual spectral radius of the MSOR method under the assumption that the eigenvalues of the iteration matrix B of the Jacobi method for system (1.3) are all purely imaginary. For this purpose we follow an analysis similar to that given in [5, pp. 277-281] for the case where all the eigenvalues of B are real. For other results in the real case see also [4].

§2. Convergence Analysis

According to Theorem 2.1 [5,p.273] for the matrix $L_{\omega,\omega'}$ the following are true: (i) If λ is an eigenvalue of $L_{\omega,\omega'}$, then there exists an eigenvalue ξ of the iteration matrix B of the Jacobi method for system (1.3) (note that

$$B = \begin{bmatrix} 0_1 & F \\ G & 0_2 \end{bmatrix}, \tag{2.1}$$

where 0_1, 0_2 are square null matrices of the same sizes as D_1 and D_2 respectively), such that

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = \omega\omega'\xi^2\lambda . \tag{2.2}$$

(ii) If ξ is a nonzero eigenvalue of B and if λ satisfies (2.2), then λ is an eigenvalue of $L_{\omega,\omega'}$. If $\xi = 0$ is an eigenvalue of B, then $\lambda = 1 - \omega$ and/or $\lambda = 1 - \omega'$ is an elgenvalue of ω, ω' .

We can write (2.2) as follows:

$$\lambda^2 - b\lambda + c = 0 , \tag{2.3}$$

where

$$c = (\omega - 1)(\omega' - 1), \quad b = \omega\omega'\xi^2 - \omega - \omega' + 2 = 1 + c - \omega\omega'(1 - \xi^2) . \tag{2.4}$$

Since $b = b(\xi^2)$, following §6.1 [5, p.170] we can define the virtual spectral radius of $L_{\omega,\omega'}$ by

$$\bar{\rho}(L_{\omega,\omega'}) = \max_{\xi^2 \in \bar{S}_{B^2}} \psi(\omega, \omega', \xi^2) , \tag{2.5}$$

where $\psi(\omega,\omega',\xi^2)$ is the root radius of (2.3), i.e. the maximum of the moduli of the roots of (2.3) and $\bar{S}_{B^2}$ is the smallest convex set containing S_{B^2}, the set of all eigenvalues of B^2. Note that, because of (2.1), we have

$$B^2 = \begin{bmatrix} FG & 0 \\ 0 & GF \end{bmatrix}. \tag{2.6}$$

If $\rho(L_{\omega,\omega'})$ is the spectral radius of $L_{\omega,\omega'}$, then $\rho(L_{\omega,\omega'}) \le \bar{\rho}(L_{\omega,\omega'})$ with equality whenever the maximum in (2.5) occurs at an eigenvalue m of B^2 and $\psi(\omega,\omega',m)$ is equal to the modulus of an eigenvalue of $L_{\omega,\omega'}$. The MSOR method is strongly convergent if and only if $\bar{\rho}(L_{\omega,\omega'}) < 1$.

In the following analysis we assume that the eigenvalues ξ of B are purely imaginary, that is we set $\xi = i\mu$, where $i^2 = -1$ and μ is real. According to [5, pp.141-145], the matrix A in (1.2) is consistently ordered and therefore iv is an eigenvalue of B if and only if $-iv$ is. Consequently, if $\sigma(B)$ is the spectrum of B, then we have

$$-\mu \le \mu \le \bar{\mu},\ \underline{\mu} \le |\mu| \le \bar{\mu} \quad \text{and} \quad \bar{S}_{B^2} \equiv [-\bar{\mu}^2, -\underline{\mu}^2],$$

where

$$\bar{\mu} = \rho(B) \text{ and } \underline{\mu} = \min_{i\mu\in\sigma(B)} |\mu| \quad (i\underline{\mu} \text{ and } i\bar{\mu} \text{ are eigenvalues of } B).$$

Thus, in our case we can replace (2.3) and (2.4) by

$$\lambda^2 - b\lambda + c = 0\ , \tag{2.7}$$

where

$$c = (\omega-1)(\omega'-1), \quad b = b(\mu^2) = -\omega\omega'\mu^2 - \omega - \omega' + 2 = 1 + c - \omega\omega'(1+\mu^2). \tag{2.8}$$

We now prove the following theorem.

Theorem 2.1. *Let A in (1.1) be a matrix of the form (1.2). If the eigenvalues of B are purely imaginary, then the MSOR method is strongly convergent if and only if*

$$0 < \omega < 2 \text{ and } 0 < \omega < \frac{2(2-\omega')}{2-\omega'(1-\bar{\mu}^2)} \equiv M(\bar{\mu}^2). \tag{2.9}$$

Proof. The MSOR method is strongly convergent if and only if the two roots of (2.7) are less than one in modulus $\forall \mu^2 \in [\underline{\mu}^2, \bar{\mu}^2]$. According to Lemma 2.1 [5, p.171], this hapens if and only if

$$|c| < 1 \text{ and } |b| < 1 + c, \quad \forall \mu^2 \in [\underline{\mu}^2, \bar{\mu}^2]. \tag{2,10}$$

Because of (2.8), (2.10) are equivalent to

$$\begin{cases} |\omega-1|\,|\omega'-1| < 1\ , \quad \omega\omega'(1+\mu^2) > 0 \quad \text{and} \\ (2-\omega)(2-\omega') - \omega\omega'\mu^2 > 0,\ \mu^2 \in [\underline{\mu}^2, \bar{\mu}^2]. \end{cases} \tag{2.11}$$

The second of (2.11) is equivalent to $\omega\omega' > 0$ and therefore in order that the third of (2.11) can be valid we must have $(2-\omega)(2-\omega') > 0$, that is $\omega, \omega' < 2$ or $\omega, \omega' > 2$. If $\omega, \omega' < 0$ or $\omega, \omega' > 2$, then the first of (2.11) is not true; hence, the only case we must consider is : $0 < \omega' < 2$ and $0, \omega' < 2$, since $\omega\omega' > 0$. If $0 < \omega < 2$ and $0 < \omega' < 2$, then the first of (2.11) is valid and the third of (2.11) is equivalent to

$$0 < \omega < \frac{2(2-\omega')}{2-\omega'(1-\mu^2)} \equiv M(\mu^2),\ \mu^2 \in [\underline{\mu}^2, \bar{\mu}^2]. \tag{2.12}$$

It can be shown that $M(\mu^2) \leq 2$ and since (2.12) must hold $\forall \mu^2 \in [\underline{\mu}^2, \bar{\mu}^2]$, we finally have

$$0 < \omega' < 2, \quad 0 < \omega < \min_{\mu^2 \in [\underline{\mu}^2, \bar{\mu}^2]} M(\mu^2) = \frac{2(2-\omega')}{2-\omega'(1-\bar{\mu}^2)}$$

i.e. (2.9).

Remarks. (i) As follows from the proof of Theorem 2.1, the roles of ω, ω' in (2.9) can be interchanged.

(ii) If $\omega = \omega'$, then (2.9) are equivalent to $\omega^2(1-\bar{\mu}^2) - 4\omega + 4 > 0$, that is $0 < \omega < \dfrac{2}{1+\bar{\mu}}$, which is a known result for the SOR method.

(iii) If $\bar{\mu} = 0$, then (2.9) reduce to $0 < \omega' < 2$ and $0 < \omega < 2$.

§3. Determination of the Optimum MSOR Method

We now study and solve the problem of determining expressions for the virtual spectral radius of $L_{\omega,\omega'}$ and of choosing ω and ω' to minimize the virtual spectral radius. For this purpose we prove the following theorem.

Theorem 3.1. *Let A in (1.1) be a matrix of the form (1.2). If the eigenvalues of B (see (2.1)) are purely imaginary, then the optimum values ω_0, ω_0' of ω, ω', respectively, of the MSOR method are the two roots of the quadratic equation:*

$$\lambda^2 - \left[\frac{2(1+r^2)}{2+\underline{\mu}^2+\bar{\mu}^2} + 1 - r^2\right]\lambda + \frac{2(1+r^2)}{2+\underline{\mu}^2+\bar{\mu}^2} = 0\ , \tag{3.1}$$

where

$$r = \frac{2+\underline{\mu}^2+\bar{\mu}^2 - 2\sqrt{(1+\underline{\mu}^2)(1+\bar{\mu}^2)}}{\bar{\mu}^2-\underline{\mu}^2} = \frac{\bar{\mu}^2-\underline{\mu}^2}{2+\underline{\mu}^2+\bar{\mu}^2 - 2\sqrt{(1+\underline{\mu}^2)(1+\bar{\mu}^2)}}$$

$$= \frac{\sqrt{1+\bar{\mu}^2} - \sqrt{1+\underline{\mu}^2}}{\sqrt{1+\bar{\mu}^2} + \sqrt{1+\underline{\mu}^2}}\ . \tag{3.2}$$

Moreover, the corresponding optimum virtual spectral radius $\bar{\rho}(L_{\omega_0,\omega_0'})$ is equal to r given by (3.2).

Proof. I. Case $\underline{\mu} = \bar{\mu}$. Then $\mu^2 = \underline{\mu}^2 = \bar{\mu}^2 \forall \mu^2$ and (2.7) becomes

$$\lambda^2 - (\omega\omega'\bar{\mu}^2 - \omega - \omega' + 2)\lambda + (\omega-1)(\omega'-1) = 0\ . \tag{3.3}$$

It is clear that in this case the optimum values of ω, ω' are those for which the two roots of (3.3) are equal to zero. Thus, from the system of equations:

$$(\omega - 1)(\omega' - 1) = 0, \quad -\omega\omega'\bar{\mu}^2 - \omega - \omega' + 2 = 0 \tag{3.4}$$

we obtain

$$\omega_0 = 1, \quad \omega_0' = \frac{1}{1+\bar{\mu}^2} \ (\text{or } \omega_0' = 1 \text{ and } \omega_0 = \frac{1}{1+\bar{\mu}^2}) \, . \tag{3.5}$$

II. Case $\underline{\mu} < \bar{\mu}$. We can by Theorem 2.1 restrict our consideration to values of ω, ω' satisfying (2.9). Following [5, pp. 277-281] we note that $\bar{\rho}(L_{\omega,\omega'})$ for fixed ω and ω' is the root radius of

$$\lambda^2 \bar{b}\lambda + c = 0 \, , \tag{3.6}$$

where

$$\bar{b} = \max_{\underline{\mu}^2 \le \mu^2 \le \bar{\mu}^2} |b(\mu^2)| = \max_{\underline{\mu}^2 \le \mu^2 \le \bar{\mu}^2} |1 + c - \omega\omega'(1+\mu^2)| \, . \tag{3.7}$$

Since $b(\mu^2)$ is a linear function of μ^2 for fixed ω and ω', we obtain by (3.7)

$$\bar{b} = \max\{|1 + c - \omega\omega'(1+\bar{\mu}^2)|, \ |1 + c - \omega\omega'(1+\underline{\mu}^2)|\} = \max\{|b(\bar{\mu}^2)|, \ |b(\underline{\mu}^2)|\} \, . \tag{3.8}$$

It can be proved that $|b(\bar{\mu}^2)| \ge |b(\underline{\mu}^2)|$ if and only if

$$\omega\omega' \ge \frac{2(1+c)}{2+\underline{\mu}^2+\bar{\mu}^2} \, , \tag{3.9}$$

or equivalently

$$\omega \ge \frac{2(2-\omega')}{2+\omega'(\underline{\mu}^2+\bar{\mu}^2)} \equiv N(\underline{\mu}^2, \bar{\mu}^2) \, . \tag{3.10}$$

Evidently, $N(\underline{\mu}^2, \bar{\mu}^2) > 0$ and we can show that

$$N(\underline{\mu}^2, \bar{\mu}^2) < M(\bar{\mu}^2) \, . \tag{3.11}$$

On the other hand it can be proved that $b(\bar{\mu}^2) \le 0$ if and only if $\omega \ge \dfrac{2-\omega'}{1+\omega'\bar{\mu}^2}$. Moreover, $b(\underline{\mu}^2) \ge 0$ if and only if $\omega \le \dfrac{2-\omega'}{1+\omega'\underline{\mu}^2}$. We can also show that the following inequalities are true:

$$\frac{2-\omega'}{1+\omega'\bar{\mu}^2} \le N(\underline{\mu}^2, \bar{\mu}^2) \le \frac{2-\omega'}{1+\omega'\underline{\mu}^2} \, . \tag{3.12}$$

Combining the above results with (2.9) we obtain

$$\bar{b} = \begin{cases} -b(\bar{\mu}^2), & \text{if } 0 < \omega < 2 \text{ and } N(\underline{\mu}^2, \bar{\mu}^2) \le \omega < M(\bar{\mu}^2), \\ b(\underline{\mu}^2), & \text{if} 0 < \omega < 2 \text{ and } 0 < \omega \le N(\underline{\mu}^2, \bar{\mu}^2). \end{cases} \tag{3.13}$$

Using (3.13) it is easy to prove that

$$\bar{b} \ge \frac{\bar{\mu}^2 - \underline{\mu}^2}{2+\underline{\mu}^2+\bar{\mu}^2}(1+c) = \frac{2r}{1+r^2}(1+c), \tag{3.14}$$

where r is given by (3.2). We have $0 < r < 1$ (note that $r = 0$ if and only if $\underline{\mu} = \bar{\mu}$). By Lemma 6–2.9 of [5] and (3.14), the root radius of (3.6) is at least as large as the root radius $\bar{\psi}(c)$ of

$$P(\lambda, c) = \lambda^2 - \frac{2r}{1+r^2}(1+c)\lambda + c = 0 . \tag{3.15}$$

We have $\bar{\psi}(r^2) = r$ and following the analysis in [5, p.280] we obtain

$$\bar{\psi}(c) > r, \text{ if } c \neq r^2 . \tag{3.16}$$

If $c = r^2$, then from (3.14) we have

$$\bar{b} \geq \frac{2r}{1+r^2}(1+r^2) = 2r \tag{3.17}$$

and the equality $\bar{b} = 2r$ holds if and only if

$$\omega\omega' = \frac{2(1+r^2)}{2+\underline{\mu}^2+\bar{\mu}^2} . \tag{3.18}$$

Hence, by Lemma 6–2.9 of [5] the roots radius of

$$Q(\lambda) = \lambda^2 - \bar{b}\lambda + r^2 = 0 \tag{3.19}$$

is greater than the root radius r of $\lambda^2 - 2r\lambda + r^2 = 0$, if

$$\omega\omega' \neq \frac{2(1+r^2)}{2+\underline{\mu}^2+\bar{\mu}^2} .$$

In order that the root radius of (3.19) shall equal r we must have $\bar{b} = 2r$ and hence (3.18) is valid. Because of $c = r^2$ and (3.18) we have

$$(\omega - 1)(\omega' - 1) = \omega\omega' - \omega\omega' + 1 = r^2 ,$$

implying that

$$\omega + \omega' = \frac{2(1+r^2)}{2+\underline{\mu}^2+\bar{\mu}^2} + 1 - r^2 . \tag{3.20}$$

Thus, from (3.18) and (3.20) we can determine the optimum values ω_0, ω_0' for the parameters ω and ω', respectively. These are two roots of equation (3.1). Moreover, we have

$$\min_{\omega,\omega'} \bar{\rho}(L_{\omega,\omega'}) = \bar{\rho}(L_{\omega_0,\omega_0'}) = r . \tag{3.21}$$

Finally, we observe that for $r = 0$, i.e. $\underline{\mu} = \bar{\mu}$, (3.1) takes the form

$$\lambda^2 - \frac{2+\bar{\mu}^2}{1+\bar{\mu}^2}\lambda + \frac{1}{1+\bar{\mu}^2} = 0 , \tag{3.22}$$

with roots $\omega_0 = 1$ and $\omega_0' = \dfrac{1}{1+\bar{\mu}^2}$; hence the optimum values obtained in case I are recovered and the proof of the theorem is complete.

§4. Comparison of the Optimum MSOR, SOR and AOR Methods

We can now compare the optimum MSOR method of Theorem 3.1 with the corresponding optimum SOR and AOR [1] methods. It is known (see e.g. [5]) that the optimum relaxation factor of the SOR method for the case of Theorem 3.1 is given by

$$\omega_b = \frac{2}{1+\sqrt{1+\bar{\mu}^2}} \tag{4.1}$$

and

$$\rho(L_{\omega_b}) = 1-\omega-b = \frac{\sqrt{1+\bar{\mu}^2}-1}{\sqrt{1+\bar{\mu}^2}+1} = \frac{2+\bar{\mu}^2-2\sqrt{1+\bar{\mu}^2}}{\bar{\mu}^2} \; . \tag{4.2}$$

Setting $x = 1+\underline{\mu}^2$, $y = 1+\bar{\mu}^2$ it is easy to show that

$$\bar{\rho}(L_{\omega_0,\omega_0'}) = \frac{\sqrt{y}-\sqrt{x}}{\sqrt{y}+\sqrt{x}} \leq \frac{\sqrt{y}-1}{\sqrt{y}+1} = \rho(L_{\omega_b}) \; ,$$

with equality if and only if $\underline{\mu} = 0$.

Let $\rho(\text{AOR})$ denote the optimum spectral radius of the AOR method for the case of Theorem 3.1. It is known (see e.g. [3]) that

$$\rho(\text{AOR}) = \begin{cases} \dfrac{\underline{\mu}\sqrt{\bar{\mu}^2-\underline{\mu}^2}}{\sqrt{1+\underline{\mu}^2}(1+\sqrt{1+\bar{\mu}^2})}, & \text{if } 1+\underline{\mu}^2 > \sqrt{1+\bar{\mu}^2} \\ 1-\omega_b \; , & \text{if } \; 1+\underline{\mu}^2 \leq \sqrt{1+\bar{\mu}^2} \; . \end{cases} \tag{4.3}$$

Consequently, if $1+\underline{\mu}^2 \leq \sqrt{1+\bar{\mu}^2}$, then $\bar{\rho}(L_{omegao,\omega_0'}) \leq \rho(\text{AOR}) = \rho(L_{\omega_b})$ with equality if and only if $\underline{\mu} = 0$. Suppose now that $1+\underline{\mu} > \sqrt{1+\mu^2}$. Obviously we have $\underline{\mu} > 0$ and thus

$$1 < x \leq y < x^2 \tag{4.4}$$

($x = 1+\underline{\mu}^2$, $y = 1+\bar{\mu}^2$). We will show that

$$\bar{\rho}(L_{\omega_0,\omega_0}) = \frac{y-x}{x+y+2\sqrt{xy}} < \frac{\sqrt{x-1}\sqrt{y-x}}{\sqrt{x}(1+\sqrt{y})} = \rho(\text{AOR}) \; , \tag{4.5}$$

or equivalently

$$(1+\sqrt{y})\sqrt{x(y-x)} < \sqrt{x-1}(x+y+2\sqrt{xy}) \; . \tag{4.6}$$

From (4.4) we have $1+\sqrt{y} < \sqrt{x}+\sqrt{y}$; hence in order to show (4.6) it is sufficient to show that

$$(\sqrt{x}\sqrt{y})\sqrt{x(y-x)} < \sqrt{x-1}(\sqrt{x}+\sqrt{y})^2 \; . \tag{4.7}$$

Then

$$\begin{aligned} (4.7) &\iff x(y-x) < (x-1)(x+y+2\sqrt{xy}) \\ &\iff (\sqrt{x}+\sqrt{y})^2 < 2x^2+2x\sqrt{xy} = 2x\sqrt{x}(\sqrt{x}+\sqrt{y}) \\ &\iff \sqrt{x}+\sqrt{y < 2x\sqrt{x}} \; . \end{aligned} \tag{4.8}$$

But, (4.8) is true since $\sqrt{x} < x\sqrt{x}$ and $\sqrt{y} < x\sqrt{x}$ (the last holds because $\sqrt{y} < x$ and $1 < \sqrt{x}$).

From the previous comparison we conclude that if $0 < \underline{\mu}$ then the optimum MSOR method is faster than the optimum AOR method. We note also that if the order of the matrix A is an odd number, then $\underline{\mu} = 0$ and the optimum MSOR method coincides with the optimum SOR method. The case $0 < \underline{\mu}$ may occur if A is of even order.

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