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# ON STABILITY AND CONVERGENCE OF THE FINITE DIFFERENCE METHODS FOR THE NONLINEAR PSEUDO-PARABOLIC SYSTEM*

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**Abstract**

In this paper, we deal with the finite difference method for the initial boundary value problem of the nonlinear pseudo-parabolic system

$$(-1)^M u_t + A(x,t,u,u_x,\ldots,u_{x^{2M-1}})u_{x^{2M}t} = F(x,t,u,u_x,\ldots,u_{x^{2M}}),$$

$$u_{x^k}(0,t) = \psi_{0k}(t),\ u_{x^k}(L,t) = \psi_{1k}(t), k = 0,1,\cdots,M-1,\ u(x,0) = \phi(x)$$

in the rectangular domain $D = [0 \le X \le L, 0 \le t \le T]$, where $u(x,t) = (u_1(x,t), u_2(x,t),\cdots,u_m(x,t))$, $\phi(x)$, $\psi_{0k}(t)$, $\psi_{1k}(t)$, $F(x,t,u,u_x,\ldots,u_{x^{2M}})$ are $m$-dimensional vector functions, and $A(x,t,u,u_x,\ldots,u_{x^{2M-1}})$ is an $m \times m$ positive definite matrix. The existence and uniqueness of solution for the finite difference system are proved by fixed-point theory. Stability , convergence and error estimates are derived.

## §1

The linear and nonlinear pseudo-parabolic equations and systems often appear in pratical research. There are many works contributed to the finite difference study of different problems for the nonlinear pseudo-parabolic equations. In this paper, we consider the nonlinear pseudo-parabolic system

$$(-1)^M u_t + A(x,t,u,u_x,\ldots,u_{x^{2M-1}})u_{x^{2M}t} = F(x,t,u,u_x,\ldots,u_{x^{2M}}) \tag{1.1}$$

with the nonhomogeneous boundary conditions

$$u_{x^k}(0,t) = \psi_{0k}(t), \quad u_{x^k}(L,t) = \psi_{1k}(t), k = 0,1,\cdots,M-1 \tag{1.2}$$

and the initial condition

$$u(x,0) = \phi(x), \tag{1.3}$$

in the rectangular domain $D = [0 \le x \le L, 0 \le t \le T]$, by the finite difference method, where $u(x,t) = (u_1(x,t), u_2(x,t),\cdots,u_m(x,t))$ and $\phi(x)$, $\psi_{0k}(t)$, $\psi_{1k}(t)$, $F(x,t,u,u_x,\ldots,u_{x^{2M}})$ are $m$-dimensional vector functions.

The equation for the long waves in nonlinear dispersion

$$u_t + f(u)_x = u_{xxt}$$

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is a simple special case of the system (1.1). Some evolutional equations of Sobolev-Galpern type also belong to the system (1.1).

We make the following assumptions for the system (1.1)-(1.3):

(1) The system (1.1)-(1.3) has a unique smooth solution.

(2) $A(x,t,p_0,p_1,\ldots,p_{2M-1})$ is an $m\times m$ symmetric positive definite matrix for $(x,t)\in D$ and $p_0,p_1,\ldots,p_{2M-1}\in \mathbb{R}^m$. Or rather, there exists a constant $a_0>0$, such that for all $\xi\in\mathbb{R}^m$,

$$(A\xi,\xi)\ge a_0|\xi|^2.$$

(3) $A(x,t,p_0,p_1,\ldots,p_{2M-1})$ is a continuous function with respect to $(x,t)\in D$ and it has the first order continuous partial derivatives with respect to $p_0,p_1,\cdots,p_{2M-1}\in\mathbb{R}^m$ and with respect to $t\in[0,T]$.

(4) $F(x,t,p_0,p_1,\ldots,p_{2M})$ is a continuous function with respect to $(x,t)\in D$ and it has the first order continuous partial derivatives with respect to $p_0,p_1,\ldots,p_{2M}\in\mathbb{R}^m$.

(5) $\psi_{ok}(t)\in C^{(1)}([0,T]),\psi_{1k}(t)\in C^{(1)}([0,T])$ and $\phi(x)\in C^{(2M)}([0,L])$ satisfy the following conditions:

$$\psi_{0k}(0)=\phi_{x^k}(0),\quad \psi_{1k}(0)=\phi_{x^k}(L),\quad k=0,1,\cdots,M-1.$$

Let us divide the rectangular domain $D$ into small grids by the parallel lines $x=x_j$ $(j=0,1,\cdots,J)$ and $t=t^n$ $(n=0,1,\cdots,N)$, where $x_j=jh,t^n=n\tau,Jh=L,N\tau=T$ $(j=0,1,\cdots,J;\ n=0,1,\cdots,N)$. Denote the vector valued discrete function on the grid point $(x_j,t^n)$ by $v_j^n$ $(j=0,1,\cdots,J;\ n=0,1,\cdots,N)$. For simplicity, we adopt the same notations and abbreviatons as used in [1-3].

Let us construct the finite difference system

$$(-1)^M\frac{v_j^{n+1}-v_j^n}{\tau}+A_j^{n+\alpha}(v)\frac{\Delta_+^M\Delta_-^M(v_j^{n+1}-v_j^n)}{\tau h^{2M}}=F_j^{n+\alpha}(v),$$
$$j=M,M+1,\cdots,J-M;\ n=0,1,\cdots,N-1 \tag{1.4}$$

where

$$A_j^{n+\alpha}(v)=A(x_j,t^{n+\alpha},\bar\delta^0v_j^{n+\alpha},\bar\delta^1v_j^{n+\alpha},\cdots\cdots,\bar\delta^{M-1}v_j^{n+\alpha},\tilde\delta^Mv_j^{n+\alpha},\cdots\cdots,\tilde\delta^{2M-1}v_j^{n+\alpha}),$$
$$F_j^{n+\alpha}(v)=F(x_j,t^{n+\alpha},\hat\delta^0v_j^{n+\alpha},\hat\delta^1v_j^{n+\alpha},\cdots\cdots,\hat\delta^{M-1}v_j^{n+\alpha},\tilde\delta^Mv_j^{n+\alpha},\cdots\cdots,\tilde\delta^{2M}v_j^{n+\alpha}),$$
$$\bar\delta^kv_j^{n+\alpha}=\sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(1)}\frac{\Delta_+^kv_i^{n+1}}{h^k}+\beta_{ki}^{(2)}\frac{\Delta_+^kv_i^n}{h^k}),k=0,1,\cdots,M-1,$$
$$\hat\delta^kv_j^{n+\alpha}=\sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(3)}\frac{\Delta_+^kv_i^{n+1}}{h^k}+\beta_{ki}^{(4)}\frac{\Delta_+^kv_i^n}{h^k}),k=0,1,\cdots,M-1,$$
$$\sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(1)}+\beta_{ki}^{(2)})=1,\ \sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(3)}+\beta_{ki}^{(4)})=1,k=0,1,\cdots,M-1,$$
$$v_j^{n+\alpha}=\alpha v_i^{n+1}+(1-\alpha)v_i^n,0\le\alpha\le1,$$
$$\tilde\delta^kv_j^{n+\alpha}=\sum_{i=j-M}^{j+M-k}\tilde\beta_{ki}\frac{\Delta_+^kv_i^{n+1}}{h^k},\ \sum_{i=j-M}^{j+M-k}\tilde\beta_{ki}=1,k=M,M+1,\cdots,2M. \tag{1.5}$$

The corresponding finite difference boundary conditions are

$$\frac{\Delta_+^k v_0^n}{h^k} = \tilde{\psi}_{0k}^n, \quad \frac{\Delta_-^k v_J^n}{h^k} = \tilde{\psi}_{1k}^n, \quad k = 0, 1, \cdots, M-1 \tag{1.6}$$

and the corresponding finite difference initial condition is

$$v_j^0 = \tilde{\phi}_j, \quad j = 0, 1, \cdots, J. \tag{1.7}$$

Suppose that the initial-boundary condition satisfies

$$\begin{aligned}
&\sum_{k=0}^{2M} \left\|\delta^k(\phi_h - \tilde{\phi}_h)\right\|_2 + \max_{n=0,1,\cdots,N} \sum_{k=0}^{M-1} \left(\left|\psi_{0k}^n - \tilde{\psi}_{0k}^n\right| + \left|\psi_{1k}^n - \tilde{\psi}_{1k}^n\right|\right) \\
&\quad + \max_{n=0,1,\cdots,N-1} \sum_{k=0}^{M-1} \Bigg[\left|\frac{\psi_{0k}^{n+1} - \psi_{0k}^n}{\tau} - \frac{\tilde{\psi}_{0k}^{n+1} - \tilde{\psi}_{0k}^n}{\tau}\right| \\
&\quad + \left|\frac{\psi_{1k}^{n+1} - \psi_{1k}^n}{\tau} - \frac{\tilde{\psi}_{1k}^{n+1} - \tilde{\psi}_{1k}^n}{\tau}\right|\Bigg] \le c_0(\tau + h)
\end{aligned} \tag{1.8}$$

where $c_0$ is independent of $\tau$ and $h$.

## §2

The following Lemmas 1–5 are easily verified by direct calculation .

**Lemma 1.** *For any $\{u_j\}$ and $\{v_j\}(j = 0, 1, \cdots, J)$, there are the relations*

$$\sum_{j=0}^{J-1} u_j \Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j - u_0 v_0 + u_J v_J, \tag{2.1}$$

$$\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- v_j = -\sum_{j=0}^{J-1} (\Delta_+ u_j)(\Delta_+ v_j) - u_0 \Delta_+ v_0 + u_J \Delta_- v_J. \tag{2.2}$$

**Lemma 2.** *For any $\{w_j\}(j = 0, 1, \cdots, J)$, there is the relation*

$$\|w_h\|_\infty \le \frac{\|w_h\|_2}{h^{1/2}}. \tag{2.3}$$

**Lemma 3.** *For any $\{w_j\}$ $(j = 0, 1, \cdots, J)$, there is the relation*

$$\|w_h\|_2^2 \le 2L^2 \|\delta w_h\|_2^2 + 4L|w_0|^2. \tag{2.4}$$

*When $\dfrac{\Delta_+^k w_0}{h^k} = \psi_{0k}$ and $\dfrac{\Delta_-^k w_J}{h^k} = \psi_{1k}$, $k = 0, 1, \cdots, M-1$, we have*

$$\|\delta^k v_h\|_2^2 \le c_1\Big(\|\delta^M v_h\|_2^2 + \sum_{k=0}^{M-1} |\psi_{0k}|^2 + \sum_{k=0}^{M-1} |\psi_{1k}|^2\Big), \quad k = 0, 1, \cdots, M-1, \tag{2.5}$$

*where $c_1$ is independent of $\tau$ and $h$.*

**Lemma 4.** *Assume that the following statement is true: for any* $w_h = \{w_n | n = 0, 1, \cdots, N\}$,

$$w_n \leq A_0 + B_0 \sum_{k=1}^{n} w_k \tau, n = 0, 1, \cdots, N, \tag{2.6}$$

*where* $A_0$ *and* $B_0$ *are constants,* $A_0 \geq 0, B_0 \geq 0$ *and* $B_0 < \frac{1}{2}$. *Then, there is*

$$\|w_h\|_\infty \leq A_0 \exp(2B_0 T). \tag{2.7}$$

**Lemma 5.** *For any discrete function*

$$w_h = \{w_j | \frac{\Delta_+^k w_0}{h^k} = \psi_{0k}, \frac{\Delta_-^k w_J}{h^k} = \psi_{1k}, k = 0, 1, \cdots, M-1, j = 0, 1, \cdots, J\}$$

*there are the relations*

$$w_k = \sum_{i=0}^{k} \binom{k}{i} h^i \psi_{0i}, k = 0, 1, \cdots, M-1, \tag{2.8}$$

$$w_{J-k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} h^i \psi_{1i}, k = 0, 1, \cdots, M-1, \tag{2.9}$$

$$\frac{\Delta_+^{k-1} w_{M-k}}{h^{k-1}} = \sum_{i=0}^{M-k} \binom{M-k}{i} h^i \psi_{0,i+k-1}, k = 1, 2, \cdots, M, \tag{2.10}$$

$$\frac{\Delta_-^{k-1} w_{J-M+1}}{h^{k-1}} = \sum_{i=0}^{M-k} (-1)^i \binom{M-k}{i} h^i \psi_{1,i+k-1}, k = 1, 2, \cdots, M. \tag{2.11}$$

**Lemma 6.** *For any discrete function* $v_h = \{v_j | j = 0, 1, \cdots, J\}$ *on the segment* $[0, L]$ *and for any given* $\epsilon > 0$, *there exists a constant* $K(\epsilon, n)$ *depending only on* $\epsilon$ *and* $n$, *such that*

$$\|\delta^k v_h\|_2 \leq \epsilon \|\delta^n v_h\|_2 + K(\epsilon, n) \|v_h\|_2, 0 \leq k \leq n, \tag{2.12}$$

*and*

$$\|\delta^k v_h\|_\infty \leq \epsilon \|\delta^n v_h\|_2 + K(\epsilon, n) \|v_h\|_2, 0 \leq k < n, \tag{2.13}$$

*where* $K(\epsilon, n)$ *is independent of* $v_h$ (see [3]).

## §3

Now we are going to prove the existence of the solution $v_j^n (j = 0, 1, \cdots, J, n = 0, 1, \cdots, N)$ for the finite difference system (1.4),(1.6),(1.7) and to get a series of a priori estimates of the solution. Assume that $u(x, t)$ is the solution of the system (1.1)-(1.3). Then, we have

$$(-1)^M \frac{u_j^{n+1} - u_j^n}{\tau} + A_j^{n+\alpha}(u) \frac{\Delta_+^M \Delta_-^M (u_j^{n+1} - u_j^n)}{\tau h^{2M}} = F_j^{n+\alpha}(u) + R, \tag{3.1}$$

$$\frac{\Delta_+^k u_0^n}{h^k} = \psi_{0k}^n + O(h), \frac{\Delta_-^k u_J^n}{h^k} = \psi_{1k}^n + O(h), k = 0, 1, \cdots, M-1, \tag{3.2}$$

$$u_j^0 = \phi_j, \tag{3.3}$$

where the truncation error $R = O(\tau + h)$.

Let $w_j^n = u_j^n - v_j^n$ ($j = 0, 1, \cdots, J$; $n = 0, 1, \cdots, N$). From (3.1), (3.2), (3.3) and (1.4), (1.6), (1.7), we get

$$
\begin{aligned}
&(-1)^M \frac{w_j^{n+1} - w_j^n}{\tau} + A_j^{n+\alpha}(v)\delta^{2M}\frac{w_j^{n+1} - w_j^n}{\tau} \\
&\quad + \sum_{k=0}^{M-1} (E_k)_j^{n+\alpha}(u, v)\bar{\delta}^k w_j^{n+\alpha}\delta^{2M}\frac{u_j^{n+1} - u_j^n}{\tau} \\
&\quad + \sum_{k=M}^{2M-1} (E_k)_j^{n+\alpha}(u, v)\tilde{\delta}^k w_j^{n+\alpha}\delta^{2M}\frac{u_j^{n+1} - u_j^n}{\tau} \\
&= \sum_{k=0}^{M-1} (G_k)_j^{n+\alpha}(u, v)\hat{\delta}^k w_j^{n+\alpha} + \sum_{k=M}^{2M} (G_k)_j^{n+\alpha}(u, v)\tilde{\delta}^k w_j^{n+\alpha} + R,
\end{aligned}
\tag{3.4}
$$

$$
\frac{\Delta_+^k w_0^n}{h^k} = \psi_{0k}^n - \tilde{\psi}_{0k}^n + O(h), \frac{\Delta_-^k w_J^n}{h^k} = \psi_{1k}^n - \tilde{\psi}_{1k}^n + O(h), k = 0, 1, \cdots, M - 1, \tag{3.5}
$$

$$
w_j^0 = \phi_j - \tilde{\phi}_j, \tag{3.6}
$$

where

$$
\begin{aligned}
(E_k)_j^{n+\alpha}(u, v) = \int_0^1 \frac{\partial A}{\partial p_k}(&x_j, t^{n+\alpha}, \lambda\bar{\delta}^0 u_j^{n+\alpha} + (1-\lambda)\bar{\delta}^0 v_j^{n+\alpha}, \cdots, \\
&\lambda\bar{\delta}^{M-1}u_j^{n+\alpha} + (1-\lambda)\bar{\delta}^{M-1}v_j^{n+\alpha}, \lambda\tilde{\delta}^M u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^M v_j^{n+\alpha}, \cdots, \\
&\lambda\tilde{\delta}^{2M-1}u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^{2M-1}v_j^{n+\alpha})\,d\lambda, \quad k = 0, 1, \cdots, 2M-1,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
(G_k)_j^{n+\alpha}(u, v) = \int_0^1 \frac{\partial F}{\partial p_k}(&x_j, t^{n+\alpha}, \lambda\hat{\delta}^0 u_j^{n+\alpha} + (1-\lambda)\hat{\delta}^0 v_j^{n+\alpha}, \cdots, \\
&\lambda\hat{\delta}^{M-1}u_j^{n+\alpha} + (1-\lambda)\hat{\delta}^{M-1}v_j^{n+\alpha}, \lambda\tilde{\delta}^M u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^M v_j^{n+\alpha}, \cdots, \\
&\lambda\tilde{\delta}^{2M}u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^{2M}v_j^{n+\alpha})d\lambda, \quad k = 0, 1, \cdots, 2M,
\end{aligned}
\tag{3.8}
$$

$$
\delta^{2M}\frac{w_j^{n+1} - w_j^n}{\tau} = \frac{\Delta_+^M \Delta_-^M (w_j^{n+1} - w_j^n)}{\tau h^{2M}}. \tag{3.9}
$$

For simplicity, we write (3.4) in the form

$$
(-1)^M \frac{w_j^{n+1} - w_j^n}{\tau} + L_2 + L_3 + L_4 = r_1 + r_2 + R, \tag{3.10}
$$

where

$$
\begin{aligned}
L_2 &= A_j^{n+\alpha}(v)\delta^{2M}\frac{w_j^{n+1} - w_j^n}{\tau}, \\
L_3 &= \sum_{k=0}^{M-1} (E_k)_j^{n+\alpha}(u, v)\bar{\delta}^k w_j^{n+\alpha}\delta^{2M}\frac{u_j^{n+1} - u_j^n}{\tau}, \\
L_4 &= \sum_{k=M}^{2M-1} (E_k)_j^{n+\alpha}(u, v)\tilde{\delta}^k w_j^{n+\alpha}\delta^{2M}\frac{u_j^{n+1} - u_j^n}{\tau},
\end{aligned}
$$

$$r_1 = \sum_{k=0}^{M-1} (G_k)_j^{n+\alpha}(u,v)\hat{\delta}^k w_j^{n+\alpha},$$
$$r_2 = \sum_{k=M}^{2M} (G_k)_j^{n+\alpha}(u,v)\tilde{\delta}^k w_j^{n+\alpha}. \tag{3.11}$$

Now we will get a series of a priori estimates for the solution of the finite difference system (3.4),(3.5),(3.6).

Making the scalar product of the vector $\delta^{2M} w_j^{n+\alpha} h\tau$ with the finite difference system (3.10) and summing up the resulting relations for $j = M, M+1, \cdots, J-M$, we get

$$\begin{aligned}(-1)^M &\sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, w_j^{n+1} - w_j^n)h + \sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, L_2)h\tau \\ &+ \sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, L_3)h\tau + \sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, L_4)h\tau \\ &= \sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, r_1)h\tau + \sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, r_2)h\tau + \sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, R)h\tau. \end{aligned} \tag{3.12}$$

First, repeating formula (2.2) and the finite difference boundary condition (3.5), we obtain

$$\begin{aligned}(-1)^M &\sum_{j=M}^{J-M} (\delta^{2M} w_j^{n+\alpha}, w_j^{n+1} - w_j^n)h \\ &= \frac{1}{2}\|\delta^M w_h^{n+1}\|_2^2 - \frac{1}{2}\|\delta^M w_h^n\|_2^2 + (\alpha - \frac{1}{2})\|\delta^M (w_h^{n+1} - w_h^n)\|_2^2 + B_1,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}B_1 = &\sum_{k=1}^{M} (-1)^{M-k} \sum_{k=1}^{M} \Big(\frac{\Delta_+^M \Delta_-^{M-k} w_{M-k}^{n+\alpha}}{h^{2M-k}}, \frac{\Delta_+^{k-1}(w_{M-k}^{n+1} - w_{M-k}^n)}{\tau h^{k-1}}\Big)\tau \\ &- \sum_{k=1}^{M} (-1)^{M-k} \sum_{k=1}^{M} \Big(\frac{\Delta_+^M \Delta_-^{M-k} w_{J-M}^{n+\alpha}}{h^{2M-k}}, \frac{\Delta_+^{k-1}(w_{J-M+1}^{n+1} - w_{J-M+1}^n)}{\tau h^{k-1}}\Big)\tau.\end{aligned} \tag{3.14}$$

Using Lemmas 5 and 6, we have

$$\begin{aligned}|B_1| \le c_2\tau\Big\{ &\|\delta^{2M} w_h^{n+\alpha}\|_2^2 + \|\delta^M w_h^{n+\alpha}\|_2^2 + \sum_{k=0}^{M-1} \Big[\Big|\frac{\psi_{0k}^{n+1} - \psi_{0k}^n}{\tau} - \frac{\tilde{\psi}_{0k}^{n+1} - \tilde{\psi}_{0k}^n}{\tau}\Big|^2 \\ &+ \Big|\frac{\psi_{1k}^{n+1} - \psi_{1k}^n}{\tau} - \frac{\tilde{\psi}_{1k}^{n+1} - \tilde{\psi}_{1k}^n}{\tau}\Big|^2\Big] + O(h^2)\Big\},\end{aligned} \tag{3.15}$$

where $c_2$ is independent of $\tau$ and $h$ .

Similarly, there is

$$\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+\alpha},L_2)h\tau=\frac{1}{2}\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+1},A_j^{n+\alpha}\delta^{2M}w_j^{n+1})h$$
$$-\frac{1}{2}\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n},A_j^{n+\alpha}\delta^{2M}w_j^{n})h$$
$$+(\alpha-\frac{1}{2})\sum_{j=M}^{J-M}(\delta^{2M}(w_j^{n+1}-w_j^n),A_j^{n+\alpha}\delta^{2M}(w_j^{n+1}-w_j^n))h. \tag{3.16}$$

Moreover, we have

$$\Big|\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+\alpha},L_3)h\tau\Big|\le c_3(\bar A)\tau[\|\delta^{2M}w_h^{n+\alpha}\|_2^2+\|\delta^M w_h^{n+1}\|_2^2+\|\delta^M w_h^n\|_2^2$$
$$+\sum_{k=0}^{M-1}(|\psi_{0k}^{n+1}-\tilde\psi_{0k}^{n+1}|^2+|\psi_{0k}^n-\tilde\psi_{0k}^n|^2+|\psi_{1k}^{n+1}-\tilde\psi_{1k}^{n+1}|^2+|\psi_{1k}^n-\tilde\psi_{1k}^n|^2)]+\tau O(h^2), \tag{3.17}$$

where $c_3(\bar A)$ is independent of $\tau$ and $h$ , and

$$\bar A=\max_{\substack{n=0,1,\cdots,N-1\\ j=0,1,\cdots,J}}\Big(\|\frac{\partial A}{\partial p_0}\|_\infty,\|\frac{\partial A}{\partial p_1}\|_\infty,\cdots,\|\frac{\partial A}{\partial p_{2M-1}}\|_\infty\Big)_j^{n+\alpha}. \tag{3.18}$$

Similarly, we have

$$\Big|\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+\alpha},-L_4+r_1+r_2+R)h\tau\Big|\le c_4(\bar A,\bar F)\tau[\|\delta^{2M}w_h^{n+\alpha}\|_2^2$$
$$+\|\delta^M w_h^{n+1}\|_2^2+\|\delta^M w_h^n\|_2^2+\sum_{k=0}^{M-1}(|\psi_{0k}^{n+1}-\tilde\psi_{0k}^{n+1}|^2+|\psi_{0k}^n-\tilde\psi_{0k}^n|^2$$
$$+|\psi_{1k}^{n+1}-\tilde\psi_{1k}^{n+1}|^2+|\psi_{1k}^n-\tilde\psi_{1k}^n|^2)+R^2]+\tau O(h^2), \tag{3.19}$$

where $c_4(\bar A,\bar F)$ is independent of $\tau$ and $h$ , and

$$\bar F=\max_{\substack{n=0,1,\cdots,N-1\\ j=0,1,\cdots,J}}\Big(\Big\|\frac{\partial F}{\partial p_0}\Big\|_\infty,\Big\|\frac{\partial F}{\partial p_1}\Big\|_\infty,\cdots,\Big\|\frac{\partial F}{\partial p_{2M}}\Big\|_\infty\Big)_j^{n+\alpha}. \tag{3.20}$$

Substituting (3.13),(3.15),(3.16),(3.17),(3.19) into the relation (3.12), we obtain

$$\frac{1}{2}\Big(\|\delta^M w_h^{n+1}\|_2^2+\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+1},A_j^{n+\alpha}(v)\delta^{2M}w_j^{n+1})h\Big)-\frac{1}{2}\Big(\|\delta^M w_h^n\|_2^2$$
$$+\sum_{j=M}^{J-M}(\delta^{2M}w_j^n,A_j^{n+\alpha}(v)\delta^{2M}w_j^n)h\Big)+(\alpha-\frac{1}{2})[\|\delta^M(w_h^{n+1}-w_h^n)\|_2^2$$

$$+\sum_{j=M}^{J-M}(\delta^{2M}(w_j^{n+1}-w_j^n),A_j^{n+\alpha}(v)\delta^{2M}(w_j^{n+1}-w_j^n))h]$$
$$\leq c_5(\bar{A},\bar{F})\tau(\|\delta^{2M}w_h^{n+\alpha}\|_2^2+\|\delta^M w_h^{n+1}\|_2^2+\|\delta^M w_h^n\|_2^2+Q^2), \tag{3.21}$$

where $c_5(\bar{A},\bar{F})$ is a constant depending only on $\bar{A}$ and $\bar{F}$ , and

$$Q^2=R^2+\sum_{k=0}^{M-1}\Big[\Big|\frac{\psi_{0k}^{n+1}-\psi_{0k}^n}{\tau}-\frac{\tilde{\psi}_{0k}^{n+1}-\tilde{\psi}_{0k}^n}{\tau}\Big|^2+\Big|\frac{\psi_{1k}^{n+1}-\psi_{1k}^n}{\tau}-\frac{\tilde{\psi}_{1k}^{n+1}-\tilde{\psi}_{1k}^n}{\tau}\Big|^2$$
$$+|\psi_{0k}^{n+1}-\tilde{\psi}_{0k}^{n+1}|^2+|\psi_{0k}^n-\tilde{\psi}_{0k}^n|^2+|\psi_{1k}^{n+1}-\tilde{\psi}_{1k}^{n+1}|^2+|\psi_{1k}^n-\tilde{\psi}_{1k}^n|^2\Big]+O(h^2). \tag{3.22}$$

We assume

$$1/2\leq\alpha\leq 1. \tag{3.23}$$

Thus, according to the assumption (2), there is

$$(\alpha-\frac{1}{2})\Big[\,\|\delta^M(w_h^{n+1}-w_h^n)\|_2^2+\sum_{j=M}^{J-M}(\delta^{2M}(w_j^{n+1}-w_j^n),A_j^{n+\alpha}(v)\delta^{2M}(w_j^{n+1}-w_j^n))h\Big]\geq 0. \tag{3.24}$$

Hence, we have

$$\frac{1}{2}(\|\delta^M w_h^{n+1}\|_2^2+\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+1},A_j^{n+\alpha+1}(v)\delta^{2M}w_j^{n+1})h)$$
$$-\frac{1}{2}(\|\delta^M w_h^n\|_2^2+\sum_{j=M}^{J-M}(\delta^{2M}w_j^n,A_j^{n+\alpha}(v)\delta^{2M}w_j^n)h)$$
$$\leq c_5(\bar{A},\bar{F})\tau(\|\delta^{2M}w_h^{n+\alpha}\|_2^2+\|\delta^M w_h^{n+1}\|_2^2+\|\delta^M w_h^n\|_2^2+Q^2)$$
$$+(\frac{\tau}{2})\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+1},\frac{A_j^{n+\alpha+1}-A_j^{n+\alpha}}{\tau}\delta^{2M}w_j^{n+1})h. \tag{3.25}$$

Let

$$\check{A}=\max_{\substack{n=0,1,\cdots,N-1\\ j=0,1,\cdots,J}}\left\{\begin{array}{l}\|\frac{\partial A}{\partial p_0}\|_\infty,\|\frac{\partial A}{\partial p_1}\|_\infty,\cdots,\|\frac{\partial A}{\partial p_{2M-1}}\|_\infty\\ \|A_j^{n+\alpha}\|_\infty,\|\frac{A_j^{n+\alpha+1}-A_j^{n+\alpha}}{\tau}\|_\infty\end{array}\right\}. \tag{3.26}$$

Then, we have

$$\frac{1}{2}(\|\delta^M w_h^{n+1}\|_2^2+\sum_{j=M}^{J-M}(\delta^{2M}w_j^{n+1},A_j^{n+\alpha+1}(v)\delta^{2M}w_j^{n+1})h)$$
$$-\frac{1}{2}(\|\delta^M w_h^n\|_2^2+\sum_{j=M}^{J-M}(\delta^{2M}w_j^n,A_j^{n+\alpha}(v)\delta^{2M}w_j^n)h)$$
$$\leq c_6(\check{A},\bar{F})\tau(\|\delta^{2M}w_h^{n+1}\|_2^2+\|\delta^{2M}w_h^n\|_2^2+\|\delta^M w_h^{n+1}\|_2^2+\|\delta^M w_h^n\|_2^2+Q^2). \tag{3.27}$$

Summing up the resulting relations (3.27) for $n = 0, 1, \cdots, l$, we get

$$\frac{1}{2}(\|\delta^M w_h^{l+1}\|_2^2 + \sum_{j=M}^{J-M}(\delta^{2M} w_j^{l+1}, A_j^{l+\alpha+1}(v)\delta^{2M} w_j^{l+1})h) - \frac{1}{2}(\|\delta^M w_h^0\|_2^2$$
$$+ \sum_{j=M}^{J-M}(\delta^{2M} w_j^0, A_j^\alpha(v)\delta^{2M} w_j^0)h) \le c_7(\check{A}, \bar{F})(\sum_{n=0}^{l+1}(\|\delta^{2M} w_h^n\|_2^2 + \|\delta^M w_h^n\|_2^2)\tau + \bar{Q}^2), \tag{3.28}$$

where $c_7(\check{A}, \bar{F})$ is a constant depending only on $\check{A}$ and $\bar{F}$ , and

$$\bar{Q} = R + \max_{n=0,1,\cdots,N} \sum_{k=0}^{M-1}\left(\left|\psi_{0k}^n - \tilde{\psi}_{0k}^n\right| + \left|\psi_{1k}^n - \tilde{\psi}_{1k}^n\right|\right)$$
$$+ \max_{n=0,1,\cdots,N-1} \sum_{k=0}^{M-1}\left(\left|\frac{\psi_{0k}^{n+1} - \psi_{0k}^n}{\tau} - \frac{\tilde{\psi}_{0k}^{n+1} - \tilde{\psi}_{0k}^n}{\tau}\right|\right.$$
$$\left.+\left|\frac{\psi_{1k}^{n+1} - \psi_{1k}^n}{\tau} - \frac{\tilde{\psi}_{1k}^{n+1} - \tilde{\psi}_{1k}^n}{\tau}\right|\right) + O(h). \tag{3.29}$$

Using the assumption (2), we have

$$\|\delta^M w_h^{l+1}\|_2^2 + \sum_{j=M}^{J-M}(\delta^{2M} w_j^{l+1}, A_j^{l+\alpha+1}(v)\delta^{2M} w_j^{l+1})h \ge c_8(\|\delta^{2M} w_h^{l+1}\|_2^2 + \|\delta^M w_h^{l+1}\|_2^2), \tag{3.30}$$

where $c_8 > 0$ is a constant independent of $\tau$ and $h$ . Hence, we have

$$\|\delta^M w_h^{l+1}\|_2^2 + \delta^{2M} w_h^{l+1}\|_2^2 \le c_9(\check{A}, \bar{F})(\sum_{n=0}^{l+1}(\|\delta^{2M} w_h^n\|_2^2$$
$$+\|\delta^M w_h^n\|_2^2)\tau + \|\delta^{2M} w_h^0\|_2^2 + \|\delta^M w_h^0\|_2^2)\tau + \bar{Q}^2), \tag{3.31}$$

where $c_9(\check{A}, \bar{F})$ is a constant depending only on $\check{A}$ and $\bar{F}$.

From Lemma 4 , when $\tau$ is sufficiently small, we get the estimate

$$\max_{n=0,1,\cdots,N}(\|\delta^M w_h^n\|_2^2 + \delta^{2M} w_h^n\|_2^2) \le c_{10}(\check{A}, \bar{F})(\|\delta^{2M} w_h^0\|_2^2 + \|\delta^M w_h^0\|_2^2) + \bar{Q}^2), \tag{3.32}$$

where $c_{10}(\check{A}, \bar{F})$ is a constant depending only on $\check{A}$ and $\bar{F}$.

From Lemmas 3 and 6, we obtain

$$\max_{n=0,1,\cdots,N} \|\delta^k w_h^n\|_2^2 \le c_{11}(\check{A}, \bar{F})(\sum_{l=0}^{2M}\|\delta^l w_h^0\|_2^2 + \bar{Q}), \quad k = 0, 1, \cdots, 2M, \tag{3.33}$$

where $c_{11}(\check{A}, \bar{F})$ is a constant depending only on $\check{A}$ and $\bar{F}$.

Now we estimate $\delta^k(\frac{w_h^{n+1} - w_h^n}{\tau})$, $k = 0, 1, \cdots, 2M$. Similarly, making the scalar product of the vector $\delta^{2M}(\frac{w_j^{n+1} - w_j^n}{\tau})h$ with the finite difference system (3.10) and summing up the resulting relations for $j = M, M+1, \cdots, J-M$, we get

$$(-1)^M \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, \frac{w_j^{n+1} - w_j^n}{\tau})h + \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, L_2)h$$
$$+ \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, L_3)h + \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, L_4)h$$
$$= \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, r_1)h + \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, r_2)h$$
$$+ \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, R)h. \tag{3.34}$$

First, using Lemma 1 and the finite difference boundary condition (3.5), we obtain

$$(-1)^M \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, \frac{w_j^{n+1} - w_j^n}{\tau})h = \|\delta^M \frac{w_h^{n+1} - w_h^n}{\tau}\|_2^2 + B_2, \tag{3.35}$$

where

$$B_2 = (-1)^{M-k} \sum_{k=1}^{M} \Big( \frac{\Delta_+^M \Delta_-^{M-k} (w_{M-k}^{n+1} - w_{M-k}^n)}{\tau h^{2M-k}}, \frac{\Delta_+^{k-1} (w_{M-k}^{n+1} - w_{M-k}^n)}{\tau h^{k-1}} \Big)$$
$$-(-1)^{M-K} \sum_{k=1}^{M} \Big( \frac{\Delta_+^M \Delta_-^{M-k} (w_{J-M}^{n+1} - w_{J-M}^n)}{\tau h^{2M-k}}, \frac{\Delta_+^{k-1} (w_{J-M+1}^{n+1} - w_{J-M+1}^n)}{\tau h^{k-1}} \Big). \tag{3.36}$$

Using Lemmas 5 and 6, we have

$$|B_2| \le c_{12}\varepsilon_1 \Big( \Big\|\delta^{2M} \frac{w_h^{n+1} - w_h^n}{\tau}\Big\|_2^2 + \Big\|\delta^M \frac{w_h^{n+1} - w_h^n}{\tau}\Big\|_2^2 \Big) + c_{13}(\varepsilon_1) \sum_{k=0}^{M-1} \Big( \Big| \frac{\psi_{0k}^{n+1} - \psi_{0k}^n}{\tau}$$
$$- \frac{\tilde\psi_{0k}^{n+1} - \tilde\psi_{0k}^n}{\tau} \Big|^2 + \Big| \frac{\psi_{1k}^{n+1} - \psi_{1k}^n}{\tau} - \frac{\tilde\psi_{1k}^{n+1} - \tilde\psi_{1k}^n}{\tau} \Big|^2 \Big) + O(h^2), \tag{3.37}$$

where $c_{12}, c_{13}$ are independent of $\tau$ and $h$ .

Moreover, we have

$$\Big| \sum_{j=M}^{J-M} (\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}, -L_3 - L_4 + r_1 + r_2 + R)h \Big|$$
$$\le (\frac{5a_0}{8}) \|\delta^{2M} \frac{w_h^{n+1} - w_h^n}{\tau}\|_2^2 + c_{14}(\bar A, \bar F)(\|\delta^{2M} w_h^{n+\alpha}\|_2^2$$
$$+ \|\delta^M w_h^{n+1}\|_2^2 + \|\delta^M w_h^n\|_2^2 + \sum_{k=0}^{M-1} \Big[ |\psi_{0k}^{n+1} - \tilde\psi_{0k}^{n+1}|^2 + \|\psi_{0k}^n - \tilde\psi_{0k}^n|^2$$
$$+ |\psi_{1k}^{n+1} - \tilde\psi_{1k}^{n+1}|^2 + |\psi_{1k}^n - \tilde\psi_{1k}^n|^2) + R^2 \Big] + O(h^2), \tag{3.38}$$

where $c_{14}$ is independent of $\tau$ and $h$.

We take $\epsilon_1$, such that

$$\epsilon_1 c_{12} \leq \frac{a_0}{8}. \tag{3.39}$$

Substituting (3.37),(3.39) into the relation (3.34), we obtain

$$\begin{aligned}&\|\delta^M \frac{w_h^{n+1}-w_h^n}{\tau}\|_2^2 + (\frac{a_0}{4})\|\delta^{2M}\frac{w_h^{n+1}-w_h^n}{\tau}\|_2^2\\ &\leq c_{15}(\bar{A},\bar{F})(\|\delta^{2M}w_h^{n+\alpha}\|_2^2 + \|\delta^M w_h^{n+1}\|_2^2 + \|\delta^M w_h^n\|_2^2 + Q^2),\end{aligned} \tag{3.40}$$

where $c_{15}(\bar{A},\bar{F})$ is a constant depending only on $\bar{A}$ and $\bar{F}$.

From (3.33), we obtain

$$\max_{n=0,1,\cdots,N}\left\|\delta^k \frac{w_h^{n+1}-w_h^n}{\tau}\right\|_2^2 \leq c_{16}(\check{A},\bar{F})\Big(\sum_{l=0}^{2M}\|\delta^l w_h^0\|_2^2 + \bar{Q}\Big),\ k=0,1,\cdots,2M, \tag{3.41}$$

where $c_{16}(\check{A},\bar{F})$ is a constant depending only on $\check{A}$ and $\bar{F}$.

Hence we have the following theorem.

**Theorem 1.** *Under conditions (1)–(5) and (3.23), assume that the solution $u$ of system (1.1)–(1.3) has $|\delta^k u_j^n| \leq \bar{M}(j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M)$, and $|\delta^k \frac{u_j^{n+1}-u_j^n}{\tau}| \leq \bar{M}(j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M)$. Then, when $\tau$ and $h$ are sufficiently small, there exists the solution $\{w_j^n\}(j=0,1,\cdots,J;n=0,1,\cdots,N)$ of system (3.4)–(3.6), such that $|\delta^k w_j^n| \leq \bar{M}(j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M)$ and $|\delta^k \frac{w_j^{n+1}-w_j^n}{\tau}| \leq \bar{M}$, $(j=0,1,\cdots,J;\ n=0,1,\cdots,N;\ k=0,1,\cdots,2M)$.*

*Proof.* For any discrete function $z=\{z_j^n|j=0,1,\cdots,J;\ n=0,1,\cdots,N\}$, we construct the following mapping $\Phi$, which maps $z\in\mathcal{R}^{m(J+1)(N+1)}$ onto $w\in\mathcal{R}^{m(J+1)(N+1)}$:

$$\begin{aligned}&(-1)^M\ \frac{w_j^{n+1}-w_j^n}{\tau} + A_j^{n+\alpha}(u-z)\delta^{2M}\frac{w_j^{n+1}-w_j^n}{\tau}\\ &+\sum_{k=0}^{M-1}(E_k)_j^{n+\alpha}(u,u-z)\bar{\delta}^k w_j^{n+\alpha}\delta^{2M}\frac{u_j^{n+1}-u_j^n}{\tau}\\ &+\sum_{k=M}^{2M-1}(E_k)_j^{n+\alpha}(u,u-z)\tilde{\delta}^k w_j^{n+\alpha}\delta^{2M}\frac{u_j^{n+1}-u_j^n}{\tau}\\ &=\sum_{k=0}^{M-1}(G_k)_j^{n+\alpha}(u,u-z)\hat{\delta}^k w_j^{n+\alpha} + \sum_{k=M}^{2M}(G_k)_j^{n+\alpha}(u,u-z)\tilde{\delta}^k w_j^{n+\alpha} + R,\end{aligned} \tag{3.42}$$

$$\frac{\Delta_+^k w_0^n}{h^k} = \psi_{0k}^n - \tilde{\psi}_{0k}^n + O(h),\ \frac{\Delta_-^k w_J^n}{h^k} = \psi_{1k}^n - \tilde{\psi}_{1k}^n + O(h),\ k=0,1,\cdots,M-1, \tag{3.43}$$

$$w_j^0 = \phi_j - \tilde{\phi}_j, \tag{3.44}$$

where

$$(E_k)_j^{n+\alpha}(u,u-z)=\int_0^1\frac{\partial A}{\partial p_k}(x_j,t^{n+\alpha},\bar{\delta}^0u_j^{n+\alpha}-(1-\lambda)\bar{\delta}^0z_j^{n+\alpha},\cdots,$$
$$\bar{\delta}^{M-1}u_j^{n+\alpha}-(1-\lambda)\bar{\delta}^{M-1}z_j^{n+\alpha},\bar{\delta}^Mu_j^{n+\alpha}-(1-\lambda)\bar{\delta}^Mz_j^{n+\alpha},\cdots,$$
$$\tilde{\delta}^{2M-1}u_j^{n+\alpha}-(1-\lambda)\tilde{\delta}^{2M-1}z_j^{n+\alpha})d\lambda,k=0,1,\cdots,2M-1,\tag{3.45}$$
$$(G_k)_j^{n+\alpha}(u,u-z)=\int_0^1\frac{\partial F}{\partial p_k}(x_j,t^{n+\alpha},\hat{\delta}^0u_j^{n+\alpha}-(1-\lambda)\hat{\delta}^0z_j^{n+\alpha},\cdots,$$
$$\hat{\delta}^{M-1}u_j^{n+\alpha}-(1-\lambda)\hat{\delta}^{M-1}z_j^{n+\alpha},\tilde{\delta}^Mu_j^{n+\alpha}-(1-\lambda)\tilde{\delta}^Mz_j^{n+\alpha},\cdots,$$
$$\tilde{\delta}^{2M}u_j^{n+\alpha}-(1-\lambda)\tilde{\delta}^{2M}z_j^{n+\alpha})d\lambda,\quad k=0,1,\cdots,2M.\tag{3.46}$$

Let

$$\Omega=\Big\{z=\{z_j^n\}\Big||\delta^kz_j^n|\le\bar{M},(j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M),$$
$$|\delta^k\frac{z_j^{n+1}-z_j^n}{\tau}|\le\bar{M},(j=0,1,\cdots,J;n=0,1,\cdots,N;\quad k=0,1,\cdots,2M)\Big\}.\tag{3.47}$$

Repeating the procedure of the proof in §3 , we can obtain

$$\max_{n=0,1,\cdots,N}\sum_{k=0}^{2M}\|\delta^kw_h^n\|_2+\max_{n=0,1,\cdots,N-1}\sum_{k=0}^{2M}\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\|_2\le K_1(\check{A},\check{F})(\sum_{k=0}^{2M}\|\delta^kw_h^0\|_2+\bar{Q}),\tag{3.48}$$

where $K_1(\check{A},\check{F})$ is a constant depending only on $\check{A}$ and $\check{F}$. When $z\in\Omega$, from the assumption (3) and (4), the relations (3.26),(3.20) can be replaced respectively by the following

$$\check{A}=\max_{\substack{(x,t)\in D\\|p_0|,\cdots,|p_{2M-1}|\le2\bar{M}}}\left\{\begin{array}{l}\|\frac{\partial A}{\partial p_0}\|_\infty,\|\frac{\partial A}{\partial p_1}\|_\infty,\cdots,\|\frac{\partial A}{\partial p_{2M-1}}\|_\infty,\\ \|A\|_\infty,\|\frac{\partial A}{\partial t}\|_\infty\end{array}\right\}\le K_2\tag{3.49}$$

and

$$\check{F}=\max_{\substack{(x,t)\in D\\|p_0|,\cdots,|p_{2M}|\le2\bar{M}}}\left(\|\frac{\partial F}{\partial p_0}\|_\infty,\|\frac{\partial F}{\partial p_1}\|_\infty,\cdots,\|\frac{\partial F}{\partial p_{2M}}\|_\infty\right)\le K_3.\tag{3.50}$$

Consequently, there are

$$\max_{n=0,1,\cdots,N}\sum_{k=0}^{2M}\|\delta^kw_h^n\|_2+\max_{n=0,1,\cdots,N-1}\sum_{k=0}^{2M}\Big\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\Big\|_2\le K_4\Big(\sum_{k=0}^{2M}\|\delta^kw_h^0\|_2+\bar{Q}\Big).\tag{3.51}$$

Here $K_2,K_3,K_4$ are independent of $h$ and $\tau$ .

From (1.8), we have

$$\max_{n=0,1,\cdots,N}\sum_{k=0}^{2M}\|\delta^kw_h^n\|_2+\max_{n=0,1,\cdots,N-1}\sum_{k=0}^{2M}\Big\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\Big\|_2\le K_5(\tau+h),\tag{3.52}$$

where $K_5$ is independent of $h$ and $\tau$ .

From Lemma 2 , we get

$$\max_{n=0,1,\cdots,N}\sum_{k=0}^{2M}\|\delta^k w_h^n\|_\infty+\max_{n=0,1,\cdots,N-1}\sum_{k=0}^{2M}\left\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\right\|_\infty\le K_5(\frac{\tau}{h^{\frac{1}{2}}}+h^{\frac{1}{2}}).\quad(3.53)$$

Taking

$$h^{\frac{1}{2}}\le\frac{\bar{M}}{2K_5},\frac{\tau}{h^{\frac{1}{2}}}\le\frac{\bar{M}}{2K_5},\quad(3.54)$$

we have the following estimates:

$$|\delta^k w_j^n|\le\bar{M},\ j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M\quad(3.55)$$

and

$$\left|\delta^k\frac{w_j^{n+1}-w_j^n}{\tau}\right|\le\bar{M},\ j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M.\quad(3.56)$$

Therefore ,

$$\Phi(\Omega)\subset\Omega.\quad(3.57)$$

In addition, $\Omega$ is a convex set and $\Phi$ is continuous. From Brouwer's theorem, the mapping $\Phi$ has at least one fixed-point $w$ , such that $\Phi w=w$. Obviously, this $\{w_j^n\}$ $(j=0,1,\cdots,J;$ $n=0,1,\cdots,N)$ is the solution of the system (3.4)–(3.6). Theorem 1 is proved .

**Corollary 1.** Under the same assumptions as in Theorem 1, the finite difference system (1.4), (1.6), (1.7) has at least one solution $\{v_j^n\}(j=0,1,\cdots,J;n=0,1,\cdots,N)$. Moreover, we have the estimates

$$|\delta^k v_j^n|\le 2\bar{M},\ j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M\quad(3.58)$$

and

$$\left|\delta^k\frac{v_j^{n+1}-v_j^n}{\tau}\right|\le 2\bar{M},\ j=0,1,\cdots,J;n=0,1,\cdots,N;k=0,1,\cdots,2M.\quad(3.59)$$

## §4

In this section we shall derive uniqueness, stability, convergence and error estimates . First, we can obtain the following uniqueness theorem.

**Theorem 2.** *Under the same assumptions as in Theorem 1, the finite difference system (1.4), (1.6), (1.7) has a unique solution* $\{v_j^n\}$ $(j=0,1,\cdots,J;n=0,1,\cdots,N)$ .

*Proof.* Assume that $\{v_j^n\}$ and $\{\bar{v}_j^n\}$ are two solutions of the finite difference system (1.4), (1.6), (1.7). Let $w_j^n=v_j^n-\bar{v}_j^n$ $(j=0,1,\cdots,J;\ n=0,1,\cdots,N)$. We can obtain

$$
\begin{aligned}
&(-1)^M \frac{w_j^{n+1}-w_j^n}{\tau} + A_j^{n+\alpha}(\bar v)\delta^{2M}\frac{w_j^{n+1}-w_j^n}{\tau} \\
&\quad + \sum_{k=0}^{M-1}(E_k)_j^{n+\alpha}(v,\bar v)\bar\delta^k w_j^{n+\alpha}\delta^{2M}\frac{v_j^{n+1}-v_j^n}{\tau} \\
&\quad + \sum_{k=M}^{2M-1}(E_k)_j^{n+\alpha}(v,\bar v)\bar\delta^k w_j^{n+\alpha}\delta^{2M}\frac{v_j^{n+1}-v_j^n}{\tau} \\
&\quad = \sum_{k=0}^{M-1}(G_k)_j^{n+\alpha}(v,\bar v)\hat\delta^k w_j^{n+\alpha} + \sum_{k=M}^{2M}(G_k)_j^{n+\alpha}(v,\bar v)\bar\delta^k w_j^{n+\alpha} + R,
\end{aligned}
\tag{4.1}
$$

$$\frac{\Delta_+^k w_0^n}{h^k} = 0, \frac{\Delta_-^k w_J^n}{h^k} = 0, k = 0,1,\cdots,M-1, \tag{4.2}$$

$$w_j^0 = 0. \tag{4.3}$$

Using Lemmas 1–6 and Corollary 1, and repeating the procedure of §3 , from (4.1)–(4.3), we have

$$\max_{n=0,1,\cdots,N}\sum_{k=0}^{2M}\|\delta^k w_h^n\|_2 + \max_{n=0,1,\cdots,N-1}\sum_{k=0}^{2M}\left\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\right\|_2 \le K_6\Big(\sum_{k=0}^{2M}\|\delta^k w_h^0\|_2 + Q\Big), \tag{4.4}$$

where $K_6$ is independent of $\tau$ and $h$ . From (4.2) and (4.3), we get

$$\bar v_j^n \equiv v_j^n,\ j = 0,1,\cdots,J; n = 0,1,\cdots,N. \tag{4.5}$$

Theorem 2 is proved .

Now we derive the following stability Theorem .

**Theorem 3.** *Under assumptions (1)–(5) and (3.26) and by choosing suitable $\tau$ and $h$, the solution of the finite difference system (1.4), (1.6), (1.7) is stable with respect to the initial boundary value $\phi$, $\psi_{0k}$, $\psi_{1k}$ and nonlinear coefficient $A$ and $F$ .*

*Proof.* Suppose $\{v_j^n\}(j = 0,1,\cdots,J; n = 0,1,\cdots,N)$ satisfies

$$
\begin{cases}
(-1)^M \dfrac{v_j^{n+1}-v_j^n}{\tau} + A_j^{n+\alpha}(v)\dfrac{\Delta_+^M\Delta_-^M(v_j^{n+1}-v_j^n)}{\tau h^{2M}} = F_j^{n+\alpha}(v), \\
\dfrac{\Delta_+^k v_0^n}{h^k} = \psi_{0k}^n, \dfrac{\Delta_-^k v_J^n}{h^k} = \psi_{1k}^n, k = 0,1,\cdots,M-1, \\
v_j^0 = \phi_j,
\end{cases}
\tag{4.6}
$$

and $\{\tilde v_j^n\}(j = 0,1,\cdots,J; n = 0,1,\cdots,N)$ satisfies

$$
\begin{cases}
(-1)^M \dfrac{\tilde v_j^{n+1}-\tilde v_j^n}{\tau} + \tilde A_j^{n+\alpha}(\tilde v)\dfrac{\Delta_+^M\Delta_-^M(\tilde v_j^{n+1}-\tilde v_j^n)}{\tau h^{2M}} = \tilde F_j^{n+\alpha}(\tilde v), \\
\dfrac{\Delta_+^k \tilde v_0^n}{h^k} = \tilde\psi_{0k}^n, \dfrac{\Delta_-^k \tilde v_J^n}{h^k} = \tilde\psi_{1k}^n, k = 0,1,\cdots,M-1, \\
\tilde v_j^0 = \tilde\phi_j.
\end{cases}
\tag{4.7}
$$

Let $w_j^n = v_j^n - \tilde{v}_j^n (j = 0, 1, \cdots, J; n = 0, 1, \cdots, N)$. Subtracting (4.7) from (4.6) yields

$$(-1)^M \frac{w_j^{n+1} - w_j^n}{\tau} + A_j^{n+\alpha}(\tilde{v})\delta^{2M} \frac{w_j^{n+1} - w_j^n}{\tau}$$
$$+ \sum_{k=0}^{M-1} (E_k)_j^{n+\alpha}(v, \tilde{v})\bar{\delta}^k w_j^{n+\alpha}\delta^{2M} \frac{v_j^{n+1} - v_j^n}{\tau}$$
$$+ \sum_{k=M}^{2M-1} (E_k)_j^{n+\alpha}(v, \tilde{v})\tilde{\delta}^k w_j^{n+\alpha}\delta^{2M} \frac{v_j^{n+1} - v_j^n}{\tau}$$
$$= \sum_{k=0}^{M-1} (G_k)_j^{n+\alpha}(v, \tilde{v})\hat{\delta}^k w_j^{n+\alpha} + \sum_{k=M}^{2M} (G_k)_j^{n+\alpha}(v, \tilde{v})\tilde{\delta}^k w_j^{n+\alpha} + R_1, \tag{4.8}$$

$$\frac{\Delta_+^k w_0^n}{h^k} = \psi_{0k}^n - \tilde{\psi}_{0k}^n, \frac{\Delta_-^k w_J^n}{h^k} = \psi_{1k}^n - \tilde{\psi}_{1k}^n, k = 0, 1, \cdots, M-1, \tag{4.9}$$

$$w_j^0 = \phi_j - \tilde{\phi}_j, \tag{4.10}$$

where

$$R_1 = (A_j^{n+\alpha}(\tilde{v}) - \tilde{A}_j^{n+\alpha}(\tilde{v}))\delta^{2M} \frac{\tilde{v}_j^{n+1} - \tilde{v}_j^n}{\tau} + F_j^{n+\alpha}(\tilde{v}) - \tilde{F}_j^{n+\alpha}(\tilde{v}). \tag{4.11}$$

Using Lemma and Corollary 1, and repeating the procedure of §3 , from (4.8)-(4.10), we have

$$\max_{n=0,1,\cdots,N} \sum_{k=0}^{2M} \|\delta^k w_h^n\|_2 + \max_{n=0,1,\cdots,N-1} \sum_{k=0}^{2M} \left\|\delta^k \frac{w_h^{n+1} - w_h^n}{\tau}\right\|_2$$
$$\leq K_7 \Big[ \sum_{k=0}^{2M} \left\|\delta^k (\phi_h - \tilde{\phi}_h)\right\|_2 + \max_{n=0,1,\cdots,N} \sum_{k=0}^{M-1} (|\psi_{0k}^n - \tilde{\psi}_{0k}^n| + |\psi_{1k}^n - \tilde{\psi}_{1k}^n|)$$
$$+ \max_{n=0,1,\cdots,N-1} \sum_{k=0}^{M-1} \left( \left| \frac{\psi_{0k}^{n+1} - \psi_{0k}^n}{\tau} - \frac{\tilde{\psi}_{0k}^{n+1} - \tilde{\psi}_{0k}^n}{\tau} \right| + \left| \frac{\psi_{1k}^{n+1} - \psi_{1k}^n}{\tau} - \frac{\tilde{\psi}_{1k}^{n+1} - \tilde{\psi}_{1k}^n}{\tau} \right| \right)$$
$$+ \max_{n=0,1,\cdots,N-1} \|A_h^{n+\alpha} - \tilde{A}_h^{n+\alpha}\|_\infty + \max_{n=0,1,\cdots,N-1} \|F_h^{n+\alpha} - \tilde{F}_h^{n+\alpha}\|_\infty \Big], \tag{4.12}$$

where $K_7$ is independent of $\tau$ and $h$. Thus, the theorem is proved.

Obviously, using the results of §3 ,we also have the following error estimates and convergence.

**Theorem 4.** *Under the same assumptions as in Theorem* 1, *we have*

$$\max_{n=0,1,\cdots,N} \|\delta^k w_h^n\|_2 \leq K_8(\tau + h), k = 0, 1, \cdots, 2M, \tag{4.13}$$

$$\max_{n=0,1,\cdots,N} \|\delta^k w_h^n\|_\infty \leq K_9(\tau + h), k = 0, 1, \cdots, 2M-1, \tag{4.14}$$

$$\max_{n=0,1,\cdots,N} \|\delta^{2M} w_h^n\|_\infty \leq K_{10}(\frac{\tau}{h^{\frac{1}{2}}} + h^{\frac{1}{2}}), \tag{4.15}$$

$$\max_{n=0,1,\cdots,N-1}\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\|_2\le K_{11}(\tau+h),k=0,1,\cdots,2M,\tag{4.16}$$

$$\max_{n=0,1,\cdots,N-1}\|\delta^k\frac{w_h^{n+1}-w_h^n}{\tau}\|_\infty\le K_12(\tau+h),k=0,1,\cdots,2M-1,\tag{4.17}$$

$$\max_{n=0,1,\cdots,N-1}\|\delta^{2M}\frac{w_h^{n+1}-w_h^n}{\tau}\|_\infty\le K_{13}(\frac{\tau}{h^{\frac{1}{2}}}+h^{\frac{1}{2}}),\tag{4.18}$$

*where $K_i$, $i=8,9,\cdots 13$, are independent of $\tau$ and $h$.*

**Corollary 2.** Under the same assumptions as in Theorem 1, the solution of the finite difference system (1.4),(1.6), (1.7) has the following convergence results: When $\tau\to 0$ and $h\to 0$ , for all $k=0,1,\cdots,2M$ we have

$$\|\delta^k(u_h-v_h)\|_2\to 0,\ n=0,1,\cdots,N,\tag{4.19}$$

$$\|\delta^k(\frac{u_h^{n+1}-u_h^n}{\tau}-\frac{v_h^{n+1}-v_h^n}{\tau})\|_2\to 0,\ n=0,1,\cdots,N-1.\tag{4.20}$$

When $\tau\to 0$ and $h\to 0$ , for all $j=0,1,\cdots,J$, and $k=0,1,\cdots,2M-1$ we have

$$|\delta^k(u_j^n-v_j^n)|\to 0,n=0,1,\cdots,N,\tag{4.21}$$

$$|\delta^k(\frac{u_j^{n+1}-u_j^n}{\tau}-\frac{v_j^{n+1}-v_j^n}{\tau})|\to 0,\ n=0,1,\cdots,N-1.\tag{4.22}$$

When $\frac{\tau}{h^{\frac{1}{2}}}\to 0$ and $h\to 0$ , for all $j=0,1,\cdots,J$, we have

$$|\delta^{2M}(u_j^n-v_j^n)|\to 0,\quad n=0,1,\cdots,N,\tag{4.23}$$

$$|\delta^{2M}(\frac{u_j^{n+1}-u_j^n}{\tau}-\frac{v_j^{n+1}-v_j^n}{\tau})|\to 0,\quad n=0,1,\cdots,N-1.\tag{4.24}$$

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