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POLYNOMIAL OF DEGREE FOUR INTERPOLATION ON TRIANGLES*

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Abstract

A method for constructing the C^1 piecewise polynomial surface of degree four on triangles is presented in this paper. On every triangle, only nine interpolation conditions, which are function values and first partial derivatives at the vertices of the triangle, are needed for constructing the surface.

§1. Introduction

In finite-element analysis, surface design and other fields, constructing an interpolation surface on triangles is often needed. In many cases, it is necessary that the surface constructed be C^1 continuous, and it is desirable that the surface constructed is a polynomial and the degree of the polynomial is as low as possible because a polynomial is simple in construction and easy to calculate. Now there are mainly three methods for constructing C^1 continuous piecewise polynomial surfaces on triangles. On every triangle, the first method needs 21 inrerpolation conditions, and the surface constructed is a polynomial of degree five. The difficulty in using the first method lies in how to find the second partial derivatives. The high degree of the surface constructed is also unfavorable. Other two methods in [1] and [2] need 12 interpolation conditions on every triangle; the surfaces constructed on every triangle are C^1 continuous piecewise cubic polynomial and quadric one respectively. The shortage of the methods in [1] and [2] is that every triangle must be subdivided. The method in [3] is suitable for constructing a C^1 polynomial surface interpolating scattered data points, but the degree of the surface constructed is seven.

The method presented in this paper needs only nine interpolation conditions on every triangle, which are function values and first partial derivatives at the vertices of the triangle. The surface constructed on every triangle is a polynomial piece of degree four. The whole C^1 surface is all composed of polynomial pieces of degree four. The limitation of the method in this paper is that the degrees of all the interior vertices of the triangulation are odd.

§2. Basic Ideas

If a vertex P_i is shared by m triangles, then the degree of P_i is m (the degree of P_i in Fig.1 is 5).

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Suppose that the degree of all vertices (except those on the boundary of the interpolation region) are odd. At every vertex $P_i (i = 1, 2, \cdots)$, the function values and first partial derivatives $\{F_i, (F_x)_i, (F_y)_i\}$ are given. The procedure of constructing an interpolation surface in this paper is as follows:

(1) On every triangle, a cubic polynomial surface piece which satisfies nine given interpolation conditions is constructed. The surface has an undefined parameter.

(2) Around every vertex of the triangulation, a C^1 continuous piecewise cubic polynomial surface piece is constructed by the surface pieces, each of which has an undefined parameter.

(3) On every triangle, a polynomial surface piece of degree four is formed by weighting the three C^1 continuous piecewise cubic polynomial surface pieces around the vertices of the triangle. The whole surface is all composed of polynomial surface pieces of degree four on the triangles.

§3. Concret Realization of the Method

(1) Construction of a surface piece having an undefined parameter on a triangle.

Suppose that T is a triangle with vertices $P_\beta = \{(x_\beta, y_\beta)\}, \beta = i, j, k$ and its number is t; see Fig.3. At the three vertices of T, there are altogether nine interpolation conditions $\{F_\beta, (F_x)_\beta, (F_y)_\beta\}$, $\beta = i, j, k$. Let $(N_t)_\beta$ be the unit normal vector of the opposite side of vertex P_β, and $((L_t)_i, (L_t)_j, (L_t)_k)$ be the area coordinates of T. Then

$$\left.\begin{aligned} (L_t)_i &= (L_t(x,y))_i = [x_j y_k - x_k y_j + (y_j - y_k)x + (x_k - x_j)y]/(2S_t), \\ (L_t)_j &= (L_t(x,y))_j = [x_k y_i - x_i y_k + (y_k - y_i)x + (x_i - x_k)y]/(2S_t), \\ (L_t)_k &= (L_t(x,y))_k = [x_i y_j - x_j y_i + (y_i - y_j)x + (x_j - x_i)y]/(2S_t), \end{aligned}\right\} \tag{3.1}$$

where S_t is the area of T.

The cubic polynomial surface piece, which has an undefined parameter, on T is

$$\begin{aligned} F_t(x,y) = {} & (F_i + A_{ij}(L_t)_j + A_{ik}(L_t)_k)(L_t)_i^2 + (F_j + A_{jk}(L_t)_k + A_{ji}(L_t)_i)(L_t)_j^2 \\ & + (F_k + A_{ki}(L_t)_i + A_{kj}(L_t)_j)(L_t)_k^2 + (A_t)_p (L_t)_i (L_t)_j (L_t)_k, \end{aligned} \tag{3.2}$$

where

$$\left.\begin{aligned} A_{ij} &= 2F_i + (F_x)_i(x_j - x_i) + (F_y)_i(y_j - y_i), \\ A_{ik} &= 2F_i + (F_x)_i(x_k - x_i) + (F_y)_i(y_k - y_i), \\ A_{jk} &= 2F_j + (F_x)_j(x_k - x_j) + (F_y)_j(y_k - y_j), \\ A_{ji} &= 2F_j + (F_x)_j(x_i - x_j) + (F_y)_j(y_i - y_j), \\ A_{ki} &= 2F_k + (F_x)_k(x_i - x_k) + (F_y)_k(y_i - y_k), \\ A_{kj} &= 2F_k + (F_x)_k(x_j - x_k) + (F_y)_k(y_j - y_k). \end{aligned}\right\} \tag{3.3}$$

It can be verified that $f_t(x,y)$ takes the nine given interpolation conditions $\{F_\beta, (F_x)_\beta, (F_y)_\beta\}$, $\beta = i, j, k$. $(A_t)_p$ is an undefined parameter.

(2) Construction of a surface piece $F_i(x,y)$ around vertex P_i.

For vertex P_i without losing generality, suppose that its degree is 5; see Fig. 2. The surface pieces corresponding to (3.2) on the five triangles are $F_1(x,y), F_2(x,y), \cdots, F_5(x,y)$

respectively; the surface piece $F_i(x,y)$ around vertex P_i should be composed of $F_i(x,y)$ ($i = 1,2,3,4,5$). In order to make the surface $F_i(x,y) \in C^1$ continuous along $P_iP_k, P_iP_l, P_iP_m, P_iP_n$ and P_iP_j, the coefficients $(A_i)_P$ ($i = 1,2,3,4,5$) should be chosen properly as follows. To determine the surface piece $F_i(x,y)$ around vertex P_i, $(A_1)_P, (A_2)_P, \cdots, (A_5)_P$ are labeled as $(A_1)_{P_i}, (A_2)_{P_i}, \cdots, (A_5)_{P_i}$ respectively.

Since (3.2) is a cubic polynomial and satisfies the given interpolation conditions at the vertices, surface pieces $F_1(x,y)$ and $F_2(x,y)$ are C^1 continuous on the common side P_iP_k as long as

$$\frac{\partial F_1(x,y)}{\partial (N_1)_j}\bigg|_{\substack{x=x'\\y=y'}} = -\frac{\partial F_2(x,y)}{\partial (N_2)_l}\bigg|_{\substack{x=x'\\y=y'}} \tag{3.4}$$

where (x',y') is the midpoint of side P_iP_k.

From (3.1) and (3.2) we have

$$\begin{aligned}\frac{\partial F_1(x,y)}{\partial (N_1)_j}\bigg|_{\substack{x=x'\\y=y'}} &= \left(F_i + \frac{A_{ik}}{2} + \frac{A_{ki}}{4}\right)\frac{\partial (L_1)_i}{\partial (N_1)_j} + \left(F_k + \frac{A_{ki}}{2} + \frac{A_{ik}}{4}\right)\frac{\partial (L_1)_k}{\partial (N_1)_j}\\ &\quad + (A_{ij} + A_{kj} + (A_1)_{P_i})\frac{\partial (L_1)_j}{\partial (N_1)_j}\Big/4,\end{aligned} \tag{3.5}$$

where

$$\left.\begin{aligned}\frac{\partial (L_1)_i}{\partial (N_1)_j} &= [(y_i - y_k)(y_j - y_k) + (x_i - x_k)(x_j - x_k)]/(2S_1D_{ik}),\\ \frac{\partial (L_1)_j}{\partial (N_1)_j} &= -D_{ik}/(2S_1),\\ \frac{\partial (L_1)_k}{\partial (N_1)_j} &= [(y_k - y_i)(y_j - y_i) + (x_k - x_i)(x_j - x_i)]/(2S_1D_{ik}),\\ D_{ik} &= ((x_i - x_k)^2 + (y_i - y_k)^2)^{1/2}.\end{aligned}\right\} \tag{3.6}$$

From (3.5) and (3.6) as well as symmetry we obtain (3.7) and (3.8):

$$\begin{aligned}\frac{\partial F_2(x,y)}{\partial (N_2)_l}\bigg|_{\substack{x=x'\\y=y'}} &= \left(F_k + \frac{A_{ki}}{2} + \frac{A_{ik}}{4}\right)\frac{\partial (L_2)_k}{\partial (N_2)_l} + \left(F_i + \frac{A_{ik}}{2} + \frac{A_{ki}}{4}\right)\frac{\partial (L_2)_i}{\partial (N_2)_l}\\ &\quad + (A_{kl} + A_{il} + (A_2)_{P_i})\frac{\partial (L_2)_l}{\partial (N_2)_l}\Big/4,\end{aligned} \tag{3.7}$$

where A_{ik} and A_{ki} are defined by (3.3) and

$$\begin{aligned}A_{kl} &= 2F_k + (F_x)_k(x_l - x_k) + (F_y)_k(y_l - y_k),\\ A_{il} &= 2F_i + (F_x)_i(x_l - x_i) + (F_y)_i(y_l - y_i),\end{aligned}$$

$$\left.\begin{aligned}\frac{\partial (L_2)_k}{\partial (N_2)_l} &= [(y_k - y_i)(y_l - y_i) + (x_i - x_k)(x_i - x_l)]/(2S_2D_{ik}),\\ \frac{\partial (L_2)_l}{\partial (N_2)_l} &= -D_{ik}/(2S_2),\\ \frac{\partial (L_2)_i}{\partial (N_2)_l} &= [(y_k - y_i)(y_k - y_l) + (x_i - x_k)(x_l - x_k)]/(2S_2D_{ik}).\end{aligned}\right\} \tag{3.8}$$

From (3.4), (3.5) and (3.7) it follows that

$$
\begin{aligned}
&\Big(F_i+\frac{A_{ik}}{2}+\frac{A_{ki}}{4}\Big)\frac{\partial(L_1)_i}{\partial(N_1)_j}+\Big(F_k+\frac{A_{ki}}{2}+\frac{A_{ik}}{4}\Big)\frac{\partial(L_1)_k}{\partial(N_1)_j}\\
&\quad+(A_{ij}+A_{kj}+(A_1)_{P_i})\frac{\partial(L_1)_j}{\partial(N_1)_j}\Big/4=-\Big[\Big(F_k+\frac{A_{ki}}{2}+\frac{A_{ik}}{4}\Big)\frac{\partial(L_2)_k}{\partial(N_2)_l}\\
&\quad+\Big(F_i+\frac{A_{ik}}{2}+\frac{A_{ki}}{4}\Big)\frac{\partial(L_2)_i}{\partial(N_2)_l}+(A_{kl}+A_{il}+(A_2)_{P_i})\frac{\partial(L_2)_l}{\partial(N_2)_l}\Big/4\Big].
\end{aligned}\tag{3.9}
$$

Substituting $\frac{\partial(L_1)_j}{\partial(N_1)_j}=-\frac{D_{ik}}{(2S_1)}$ and $\frac{\partial(L_2)_l}{\partial(N_2)_l}=-\frac{D_{ik}}{(2S_2)}$ (from (3.6) and (3.8)) in (3.9), we obtain

$$
\begin{aligned}
&\frac{(A_1)_{P_i}}{S_1}+\frac{(A_2)_{P_i}}{S_2}=8\Big[\Big(F_i+\frac{A_{ik}}{2}+\frac{A_{ki}}{4}\Big)\Big(\frac{\partial(L_1)_i}{\partial(N_1)_j}+\frac{\partial(L_2)_i}{\partial(N_2)_l}\Big)\\
&\quad+\Big(F_k+\frac{A_{ki}}{2}+\frac{A_{ik}}{4}\Big)\Big(\frac{\partial(L_1)_k}{\partial(N_1)_j}+\frac{\partial(L_2)_k}{\partial(N_2)_l}\Big)\Big]\Big/D_{i,k}\\
&\quad=-\frac{A_{ij}+A_{kj}}{S_1}-\frac{(A_{kl}+A_{il})}{S_2}.
\end{aligned}\tag{3.10}
$$

Let $G(i,j,k,l)$ denote the right-hand side of (3.10); then (3.10) can be written as

$$
(A_1)_{P_i}/S_1+(A_2)_{P_i}/S_2=G(i,j,k,l).\tag{3.11}
$$

Similarly, we may obtain the other four equations to make the surface piece $F_i(x,y)$ continuous along every side with a vertex P_i of the five triangles

$$
\left.\begin{aligned}
&(A_2)_{P_i}/S_2+(A_3)_{P_i}/S_3=G(i,k,l,m),\\
&(A_3)_{P_i}/S_3+(A_4)_{P_i}/S_4=G(i,l,m,n),\\
&(A_4)_{P_i}/S_4+(A_5)_{P_i}/S_5=G(i,m,n,j),\\
&(A_5)_{P_i}/S_5+(A_1)_{P_i}/S_1=G(i,n,j,k).
\end{aligned}\right\}\tag{3.12}
$$

The coefficient matrix of the simultaneous equations (3.11) and (3.12) is nonsingular; the solution of (3.11) and (3.12) is

$$
\begin{bmatrix}(A_1)_{P_i}/S_1\\(A_2)_{P_i}/S_2\\(A_3)_{P_i}/S_3\\(A_4)_{P_i}/S_4\\(A_5)_{P_i}/S_5\end{bmatrix}=\begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\1&0&0&0&1\end{bmatrix}^{-1}\begin{bmatrix}G(i,j,k,l)\\G(i,k,l,m)\\G(i,l,m,n)\\G(i,m,n,j)\\G(i,n,j,k)\end{bmatrix}.\tag{3.13}
$$

(3.4)-(3.13) illustrate that if the undefined parameters of the five surface pieces on triangles in Fig.2 are defined by (3.13), then the surface piece $F_i(x,y)$ composed of the five surface pieces around the vertex P_i is C^1 continuous.

For every vertex P_β on the boundary of the triangulation, undefined parameters of surface piece $F_\beta(x,y)$ around P_β cannot be defined solely because of lack of one equation.

For this case, one solution is to add one equation according to the concrete case. Another method is that the surface pieces around boundary vertices are defined by those around interior vertices. For example, in Fig.3, suppose that P_4 is a boundary vertex. Surface piece $F_4(x,y)$ should be composed of surface pieces on three triangles whose numbers are 1, 2 and 3 respectively. The undefined parameter $(A_2)_{P_4}$ may be $((A_2)_{P_1}+(A_2)_{P_2})/2$. When $(A_2)_{P_4}$ is defined, undefined parameters $(A_1)_{P_4}$ and $(A_3)_{P_4}$ may be defined by (3.11) and the first equation of (3.12).

(3) Construction of a surface piece on a triangle.

For every triangle T, whose number is t and vertices are P_i, P_j and P_k (see Fig.1), the surface piece $E_t(x,y)$ on T is defined as follows:

$$E_t(x,y) = (L_t)_i F_i(x,y) + (L_t)_j F_j(x,y) + (L_t)_k F_k(x,y), \tag{3.14}$$

where $(L_t)_i, (L_t)_j$ and $(L_t)_k$ are defined by (3.1), and $F_i(x,y), F_j(x,y)$ and $F_k(x,y)$, around P_i, P_j and P_k respectively, are defined by the previous method. From (3.2) and the construction procedure of $F_i(x,y)$, $F_j(x,y)$ and $F_k(x,y)$, it is easy to know that on T, $F_i(x,y), F_j(x,y)$ and $F_k(x,y)$ are respectively

$$\left.\begin{aligned}
F_i(x,y) &= f(x,y) + (A_t)_{P_i}(L_t)_i(L_t)_j(L_t)_k,\\
F_j(x,y) &= f(x,y) + (A_t)_{P_j}(L_t)_i(L_t)_j(L_t)_k,\\
F_k(x,y) &= f(x,y) + (A_t)_{P_k}(L_t)_i(L_t)_j(L_t)_k,
\end{aligned}\right\} \tag{3.15}$$

where

$$\begin{aligned}
f(x,y) = {} & (F_i + A_{ij}(L_t)_j + A_{ik}(L_t)_k)(L_t)_i^2 + (F_j + A_{jk}(L_t)_k + A_{ji}(L_t)_i)(L_t)_j^2\\
& +(F_k + A_{ki}(L_t)_i + (A_{kj})(L_t)_j)(L_t)_k^2.
\end{aligned}$$

From (3.15) and $(L_t)_i + (L_t)_j + (L_t)_k = 1$, it is obvious that (3.14) can be written as

$$E_t(x,y) = f(x,y) + [(L_t)_i(A_t)_{pi} + (L_t)_j(A_t)_{pj} + (L_t)_k(A_t)_{pk}](L_t)_i(L_t)_j(L_t)_k \tag{3.16}$$

$E_t(x,y)$ is a polynomial of degree four because $(L_t)_i, (L_t)_j$ and $(L_t)_k$ are linear polynomials.

A whole interpolation surface $E(x,y)$ is defined as

$$E(x,y) = E_t(x,y) \qquad \text{when } (x,y) \in \text{ triangle } T. \tag{3.17}$$

We are now going to study the continuity of $E(x,y)$. For the surface piece $E_t(x,y)$ from (3.14), there are

$$E_t(x,y)\big|_{(L_t)_j=0} = \{(L_t)_i F_i(x,y) + (L_t)_k F_k(x,y)\}\big|_{(L_t)_j=0} \tag{3.18}$$

and

$$\begin{aligned}
\frac{\partial E_t(x,y)}{\partial (N_t)_j}\bigg|_{(L_t)_j=0} = {} & \Big\{F_i(x,y)\frac{\partial (L_t)_i}{\partial (N_t)_j} + (L_t)_i\frac{\partial F_i(x,y)}{\partial (N_t)_j} + F_j(x,y)\frac{\partial (L_t)_j}{\partial (N_t)_j}\\
& +F_k(x,y)\frac{\partial (L_t)_k}{\partial (N_t)_j} + (L_t)_k\frac{\partial F_k(x,y)}{\partial (N_t)_j}\Big\}_{(L_t)_j=0}
\end{aligned} \tag{3.19}$$

on the side $(L_t)_j = 0$. Since $F_i(x,y), F_j(x,y)$ and $F_k(x,y)$ have the same function values (see (3.15)) on the side $(L_t)_j = 0$, and $(L_t)_i + (L_t)_j + (L_t)_k = 1$, then

$$\left\{F_i(x,y)\frac{\partial(L_t)_i}{\partial(N_t)_j} + F_j(x,y)\frac{\partial(L_t)_j}{\partial(N_t)_j} + F_k(x,y)\frac{\partial(L_t)_k}{\partial(N_t)_j}\right\}_{(L_t)_j=0}$$
$$= \left\{F_i(x,y)\frac{\partial}{\partial(N_t)_j}((L_t)_i + (L_t)_j + (L_t)_k)\right\}_{(L_t)_j=0} = 0.$$

So (3.19) can the written as

$$\left.\frac{\partial E_t(x,y)}{\partial(N_t)_j}\right|_{(L_t)_j=0} = \left\{(L_t)_i\frac{\partial F_i(x,y)}{\partial(N_t)_j} + (L_t)_k\frac{\partial F_k(x,y)}{\partial(N_t)_j}\right\}\bigg|_{(L_t)_j=0}. \tag{3.20}$$

For triangles showed in FIg.4, let $((L_1)_i, (L_1)_j, (L_1)_k)$ and $((L_2)_i, (L_2)_k, (L_2)_n)$ (see (3.1)) be the area coordinates of the two triangles respectively and (x,y) be a point on the common side P_iP_k of the two triangles. Then

$$\left.\begin{aligned}(L_1(x,y))_i &= (L_2(x,y))_i,\\ (L_1(x,y))_k &= (L_2(x,y))_k.\end{aligned}\right\} \tag{3.21}$$

(3.18), (3.20) and (3.21) illustrate that if every surface piece on a triangle is defined by (3.14), then two surface pieces on neighbouring triangles have the same function values and first normal derivatives on the common side.

Therefore the following theorem follows:

Theorem 1. *The interpolation surface $E(x,y)$ defined by (3.17) is C^1 continuous.*

It is easy to prove the following theorem:

Theorem 2. *The polynomial precision set of (3.17) is*

$$\{x^3, x^2y, xy^2, y^3, xy, y^2, x, y, 1\}. \tag{3.22}$$

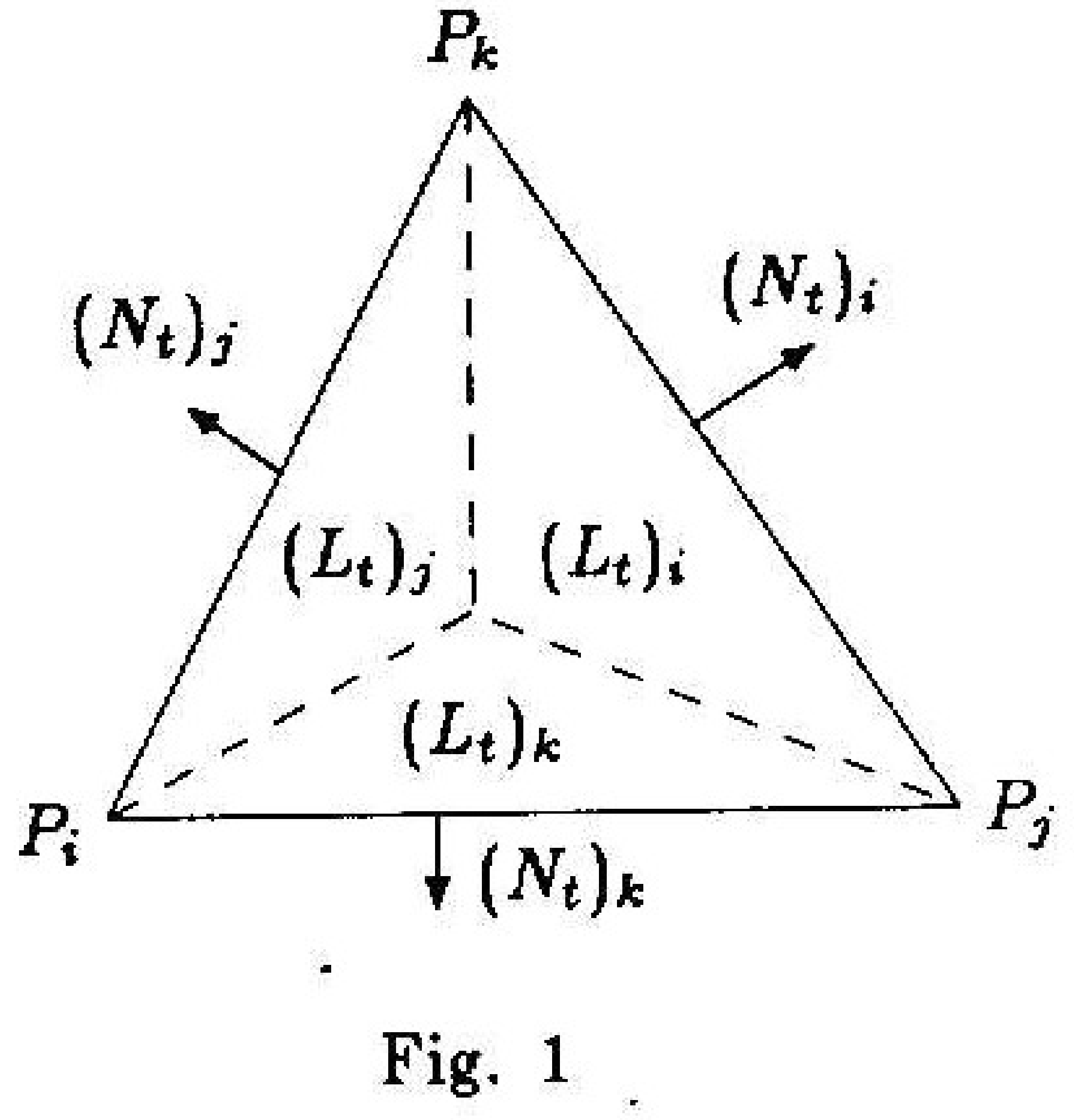

Fig. 1

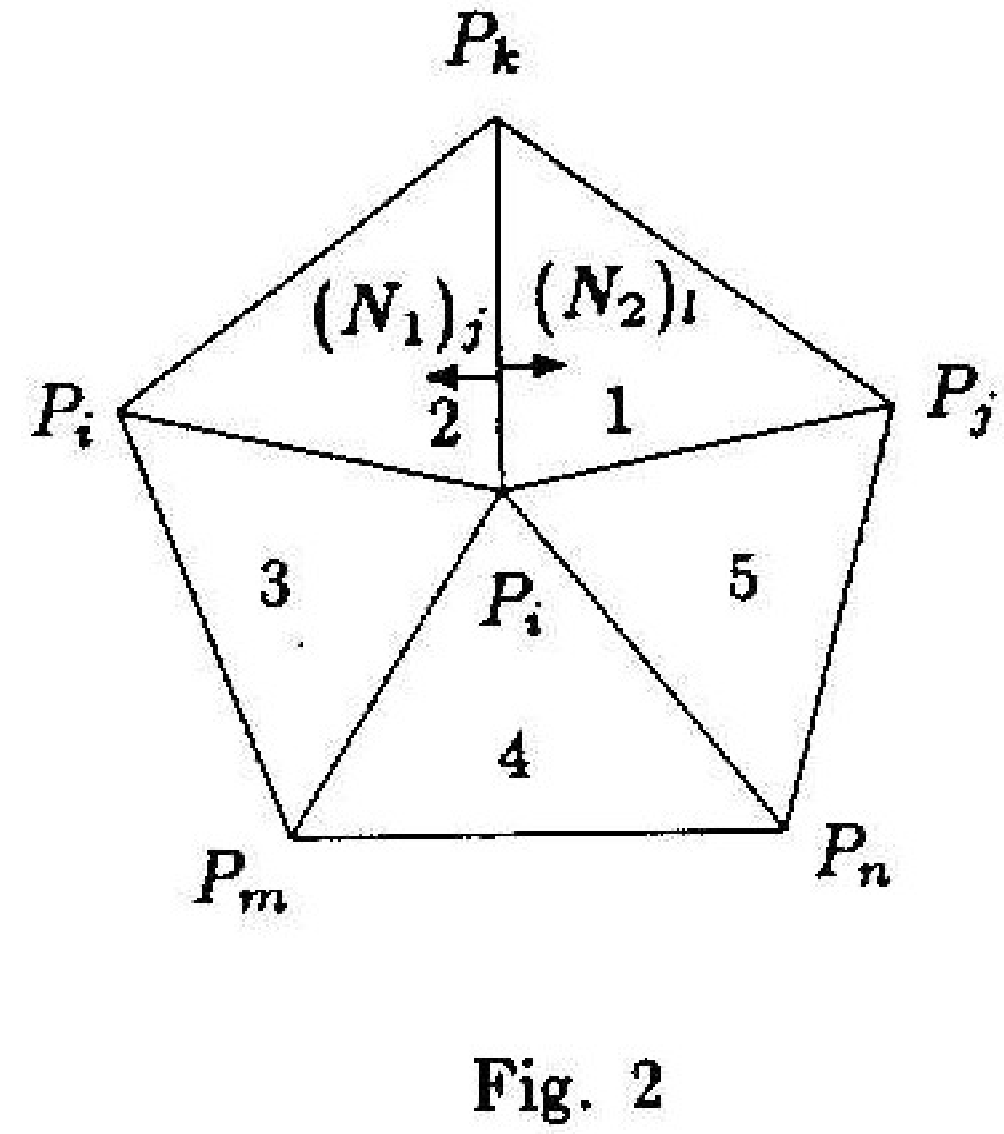

Fig. 2

§4. Triangulation Satisfying all the interior Vertices Having Odd Degree

There are two steps to reach the goal. The first step is to triangulate the given points. The second one is to modify all interior vertices of even degree into odd degree. The principle of modification is that if the degree of a vertex P_i to be modified is $2m$, then the degree of P_i is modified into $2m-1$ or $2m+1$, and the smallest angle of new triangles obtained by modification is made as large as possible. As an example, for a vertex P_i of degree (see Fig.5), its degree can be modified into 5 (as modification A) or 7 (as modification B). The modifications for all the interior vertices of even degree can be completed by a procedure from inside to outside.

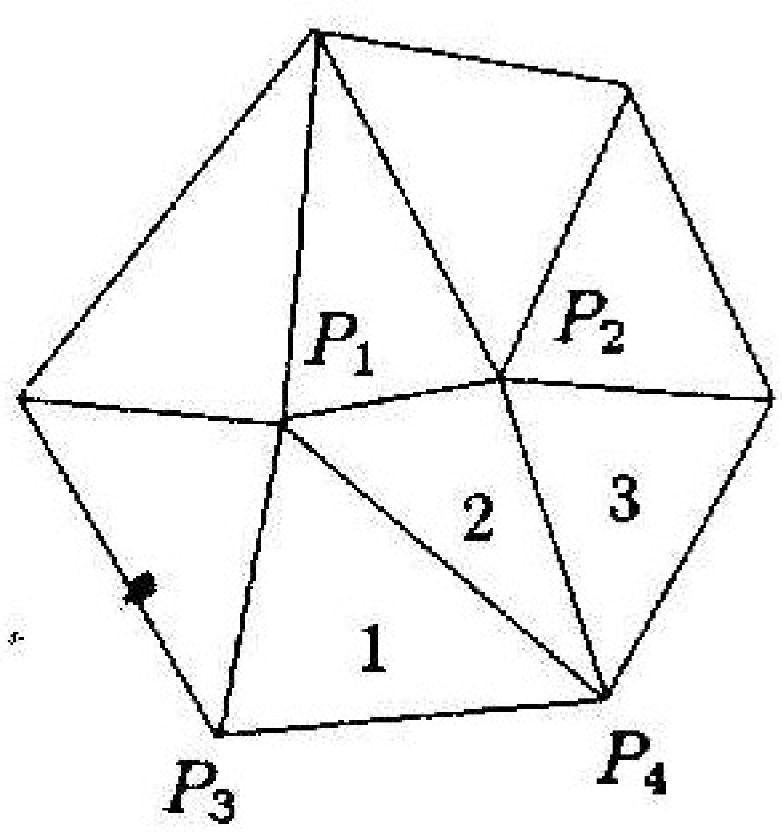

Fig. 3

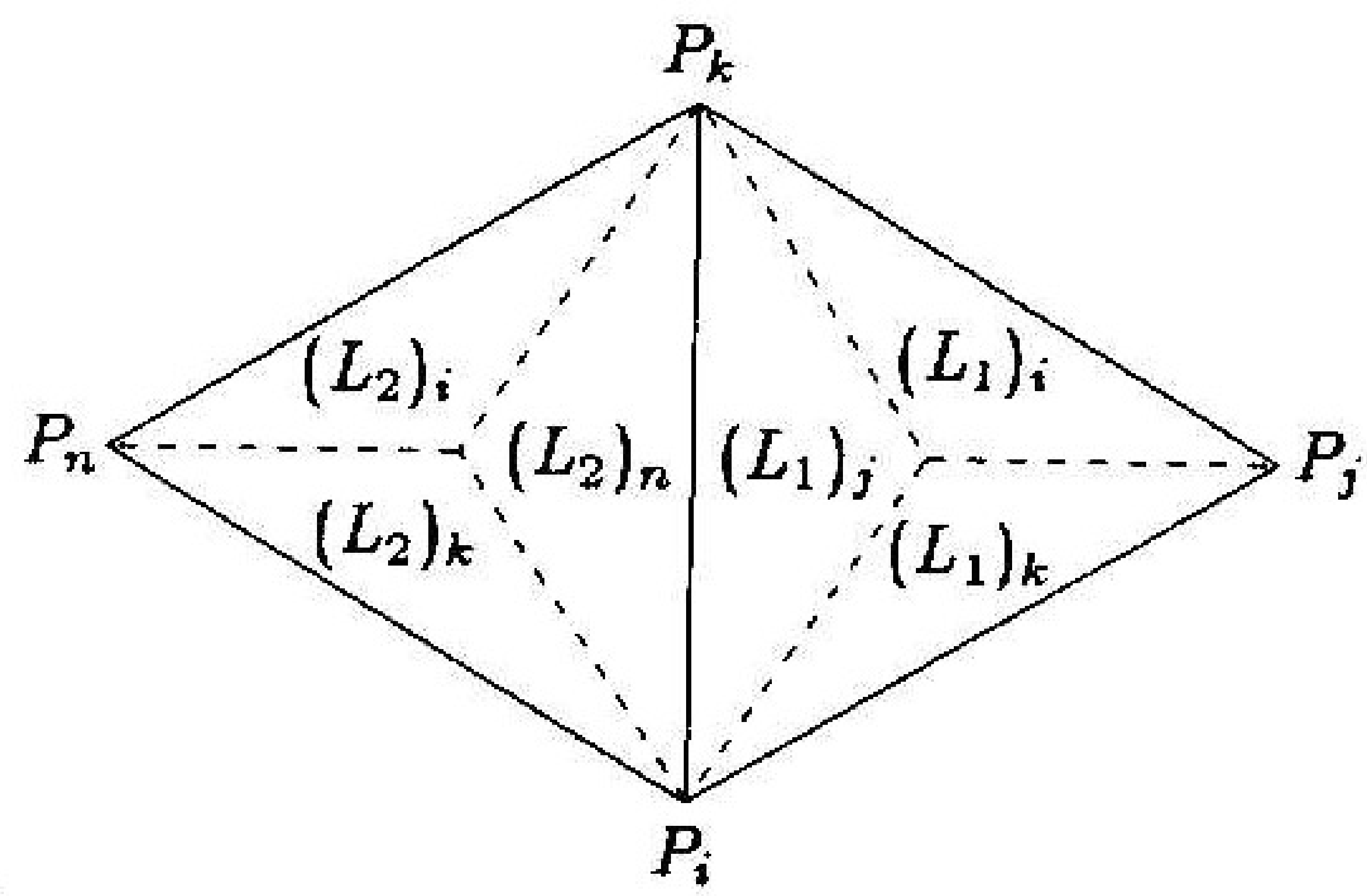

Fig. 4

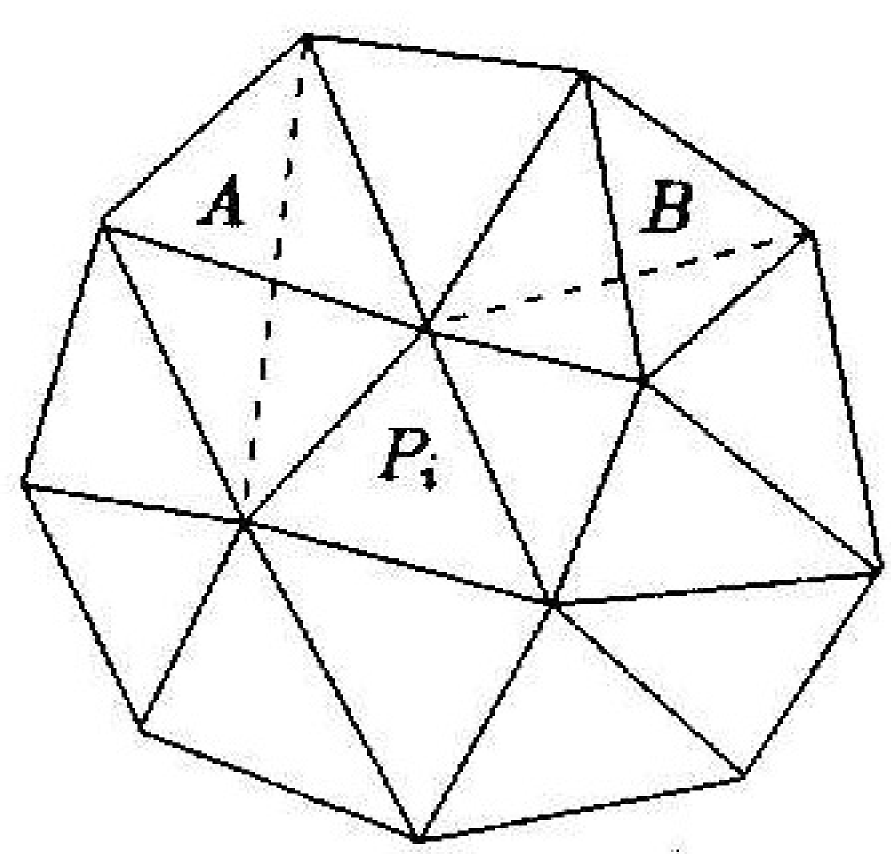

Fig. 5

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