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SOLVING BOUNDARY VALUE PROBLEMS FOR THE MATRIX EQUATION $X^{(2)}(t) - AX(t) = F(t)$ *

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Abstract

In this paper we present a method for solving the matrix differential equation $X^{(2)}(t) - AX(t) = F(t)$, without increasing the dimension of the problem. By introducing the concept of co-square root of a matrix, existence and uniqueness conditions for solutions of boundary value problems related to the equation as well as explicit solutions of these solutions are given, even for the case where the matrix A has no square roots.

§1. Introduction

Second order matrix differential equations with constant coefficients appear in the theory of damped systems and vibrational systems[5,7]. Explicit formulas for solutions of the matrix differential equation

$$X^{(2)} - AX = 0 \tag{1.1}$$

have been obtained in [1], but such formulas are not helpful for solving Cauchy problems and boundary value problems related to the non-homogeneous matrix equation

$$X^{(2)}(t) - AX(t) = F(t). \tag{1.2}$$

In a recent paper [9] we studied the boundary value problem

$$\begin{aligned} &X^{(2)}(t) - AX(t) = 0, \quad 0 \le t \le a, \\ &E_1 X(0) + E_2 X^{(1)}(0) = 0, \quad F_1 X(a) + F_2 X^{(1)}(a) = 0 \end{aligned} \tag{1.3}$$

where $E_i, F_i, i = 1, 2, A$ and $X(t)$ are $n \times n$ complex matrices. Under the invertibility hypothesis of matrix A, conditions for the existence of non-trivial solutions of (1.3) and explicit expressions of these solutions in terms of appropriate square roots of A are given.

In this paper we extend the concept of square root of a square matrix A. Hence we get an interesting expression for the general solution of equation (1.1) that allows us to obtain a generalized variation of the parameter method for the matrix equation (1.2). Finally, from the expression of the general solution of equation (1.2), existence and uniqueness conditions for solutions of the problem

$$\begin{aligned} &X^{(2)}(t) - AX(t) = F(t), \quad 0 \le t \le a, \\ &E_1 X(0) + E_2 X^{(1)}(0) = 0, \quad F_1 X(a) + F_2 X^{(1)}(a) = 0 \end{aligned} \tag{1.4}$$

as well as an explicit expression of these solutions are obtained.

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The paper is organized as follows. In Section 2 we introduce the concept of co-square root of a complex matrix A and the concept of a fundamental pair of co-square roots of a matrix. An easy characterization of this last concept and a method for obtaining fundamental pairs of co-square roots are presented. Section 3 is concerned with problems (1.2) and (1.4).

§2. Co-Square Roots of Matrices

We begin this section with the concept of co-square root of a matrix A, that is a generalization of the well-known concept of square root of a matrix. This concept will be used below to solve problems (1.2) and (1.4), and it is interesting because there are matrices without square roots[4].

In the following the set of all $n \times n$ complex matrices will be denoted by $\mathbb{C}_{n\times n}$. If A is a matrix in $\mathbb{C}_{n\times n}$, we represent by $\sigma(A)$ the set of eigenvalues of A. For a rectangular complex matrix N we will represent by N^+ the Penrose-Moore pseudoinverse of N. An account of properties and applications of this concept may be found in [10].

Definition 2.1. *Let A be a matrix in $\mathbb{C}_{n\times n}$. We say that a pair of matrices, $(X,T) \in \mathbb{C}_{n\times n} \times \mathbb{C}_{n\times n}$, is a co-square root of A, if $X \neq 0$ and*

$$XT^2 - AX = 0. \tag{2.1}$$

Example 1. If $A \in \mathbb{C}_{n\times n}$ and B is a square root of A, and I denotes the identity matrix in $\mathbb{C}_{n\times n}$, then (I, B) is a co-square root of A.

Example 2. Let z be an eigenvalue of A, and let w be a complex number such that $w^2 = z$; then the kernel of $(w^2 I - A)$ is nontrivial. Thus, for any nonzero matrix $X \in \mathbb{C}_{n\times n}$ such that $(w^2 I - A)X = 0$, the pair (X, wI) is a co-square root of A.

Definition 2.2. *Let $A \in \mathbb{C}_{n\times n}$, and let (X_i, T_i) for $i = 1,2$ be co-square roots of A. We say that the pair $\{(X_1,T_1),(X_2,T_2)\}$ is a fundamental system of co-square roots of A, if the block partitioned matrix*

$$V = \begin{bmatrix} X_1 & X_2 \\ X_1T_1 & X_2T_2 \end{bmatrix} \tag{2.2}$$

in invertible in $\mathbb{C}_{2n\times 2n}$.

Example 3. Let $A \in \mathbb{C}_{n\times n}$, and let us suppose that T_1, T_2 is a pair of square roots of A; then the pair $\{(I,T_1),(I,T_2)\}$ is a fundamental system of co-square roots of A, if and only if the matrix $T_2 - T_1$ is invertible; see lemma 1 of [9] for details.

The next result provides a characterization for the existence of a fundamental system of co-square roots of a matrix A, and it shows that for a very general class of matrices $A \in \mathbb{C}_{n\times n}$ the construction of a fundamental pair of co-square roots is available.

Theorem 1. *Let $A \in \mathbb{C}_{n\times n}$ and let $C_L = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$. Then A admits a fundamental system of co-square roots, if and only if the matrix C_L is similar to a block diagonal matrix J of the form $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, where $J_i \in \mathbb{C}_{n\times n}$ for $i = 1,2$. In this case, if $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and $PJ = C_L P$, then the pair $\{(P_{11},J_1),(P_{12},J_2)\}$ defines a fundamental system of co-square roots of A.*

Proof. Let us suppose that $PJ = C_L P$, where $P = (P_{ij})$ is an invertible block partitioned matrix with $P_{ij} \in \mathbb{C}_{n\times n}$, for $1 \le i,\ j \le 2$. From the equality

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{2.3}$$

we have

$$P_{1i} J_i = P_{2i}, \quad P_{2i} J_i = A P_{1i}, \quad i = 1, 2. \tag{2.4}$$

From (2.4) we have $P_{1i} J_i^2 = A P_{1i}$ for $i = 1, 2$. Thus (P_{11}, J_1) and (P_{12}, J_2) define a pair of co-square roots of A, because from the invertibility of P and (2.4) one gets $P_{11} \neq 0, P_{12} \neq 0$. Also note that $\{(P_{11}, J_1), (P_{12}, J_2)\}$ is a fundamental set of co-square roots of A because the corresponding matrix V given by (2.2) takes the form

$$V = \begin{bmatrix} P_{11} & P_{12} \\ P_{11} J_1 & P_{12} J_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = P.$$

Conversely, let us suppose that $\{(X_1, T_1), (X_2, T_2)\}$ is a fundamental system of co-square roots of A. Then from Definition 2.2, the matrix V defined by (2.2) is invertible and an easy computation yields

$$V[\mathrm{Diag}(T_1, T_2)] = C_L V.$$

Hence the result is proved.

Let C_L be the companion matrix introduced in Theorem 1. An easy computation shows that $\sigma(C_L) = \{z \in \mathbb{C};\, z^2 \in \sigma(A)\}$. The next lemma will be used to find concrete conditions in terms of matrix A, in order to know when the matrix C_L satisfies the condition of Theorem 1.

Lemma 1. *Let $A \in \mathbb{C}_{n\times n}$, and let $w \in \sigma(C_L)$. Then for any positive integer $q \ge 1$ the following equality holds:*

$$\dim \ker(C_L - wI)^q = \dim \ker(C_L + wI)^q. \tag{2.5}$$

Proof. Let x_1, x_2 be vectors in $\mathbb{C}_{n\times l}$. First of all we will prove by induction that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \ker(C_L - wI)^q$ if and only if $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} \in \ker(C_L + wI)^q$. If $q = 1$, then the condition $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \ker(C_L - wI)$ is equivalent to the condition

$$-wx_1 + x_2 = 0, \quad Ax_1 - wx_2 = 0. \tag{2.6}$$

Now, condition (2.6) means that $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} \in \ker(C_L + wI)$.

Let us suppose that for $q-1$ the above property is satisfied, and let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \ker(C_L - wI)^q$. Then we have

$$\begin{bmatrix} -wI & I \\ A & -wI \end{bmatrix}^q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -wI & I \\ A & -wI \end{bmatrix}^{q-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0, \tag{2.7}$$

where

$$y_1 = -wx_1 + x_2; \quad y_2 = Ax_1 - wx_2. \tag{2.8}$$

By application of the induction hypothesis and from (2.7)–(2.8), we have that (2.7) is equivalent to the condition

$$\begin{bmatrix} -y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} wx_1 - x_2 \\ Ax_1 - wx_2 \end{bmatrix} \in \ker(C_L + wI)^{q-1}. \tag{2.9}$$

Condition (2.9) means that

$$\begin{bmatrix} wI & I \\ A & wI \end{bmatrix}^q \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = (C_L + wI)^q \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = 0.$$

Now, considering that the application $\Psi : \mathbb{C}_{2n\times l} \to \mathbb{C}_{2n\times l}$, defined by $\Psi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$, is an isomorphism in $\mathbb{C}_{2n\times l}$, the result is concluded.

Theorem 2. *Let C_L be the companion matrix of Lemma* 1.

(i) *If $w \neq 0, w \in \sigma(C_L)$, then for $q \geq 1$ the following equality holds:*

$$\dim\ker(C_L - wI)^q = \dim\ker(A - w^2 I)^q. \tag{2.10}$$

(ii) *If p is a positive integer, then*

$$\begin{aligned} \dim\ker C_L^{2p+1} - \dim\ker C_L^{2p} &= \dim\ker C_L^{2p+2} - \dim\ker C_L^{2p+1} \\ &= \dim\ker A^{p+1} - \dim\ker A^p. \end{aligned} \tag{2.11}$$

Proof. (i) An easy computation yields that $C_L^2 = [\mathrm{Diag}(A, A)]$ and hence we have

$$(C_L^2 - w^2 I)^q = \mathrm{Diag}[(A - w^2 I)^q, (A - w^2 I)^q], \quad q \geq 1. \tag{2.12}$$

From (2.12) we get

$$\begin{aligned} \dim\ker(C_L^2 - w^2 I)^q &= \dim\ker(C_L + wI)^q + \dim\ker(C_L - wI)^q \\ &= 2(n - \mathrm{rang}(A - w^2 I)^q). \end{aligned} \tag{2.13}$$

From (2.13) and Lemma 1, it follows that

$$2\dim\ker(C_L - wI)^q = 2\dim\ker(A - w^2 I)^q$$

and hence (i) is proved.

(ii) It is sufficient to consider the following relationships:

$$C_L^{2p} = \begin{bmatrix} A^p & 0 \\ 0 & A^p \end{bmatrix}, \quad C_L^{2p+1} = \begin{bmatrix} 0 & A^p \\ A^{p+1} & 0 \end{bmatrix}, \qquad \text{for } p \geq 0.$$

Remark 1. From Theorem 2 we have all the information about the Jordan blocks of the Jordan canonical form of the matrix C_L, in terms of the corresponding Jordan blocks of the matrix A. In fact, for each Jordan block of A with dimension the one corresponding to the eigenvalue $z \in \sigma(A), z \neq 0$, we have two Jordan blocks for the matrix C_L with the same dimension and corresponding to the eigenvalues $-z^{1/2}, z^{1/2}$ of C_L. The nilpotent Jordan blocks of A, this is, those corresponding to the eigenvalue $z = 0$, provide Jordan blocks for C_L with dimension twice the dimension of the Jordan blocks of A.

In order to construct a fundamental system of co-square roots of A, this is, in order to determine P_{11} and P_{12}, we can find a Jordan basis of C_L, whose vectors define the columns of P. Also, by application of Kronecker products[8], and since the blocks J_i^2, appearing in Theorem 1, for $i = 1, 2$, are upper triangular, the algebraic equations

$$P_{1i}J_i^2 = AP_{1i}, \quad i = 1, 2$$

may be solved in an easy way from the knowledge of the blocks J_i, for $i = 1, 2$; see [8], chapter 12, for details.

§3. On the Matrix Differential Equation $X^{(2)} - AX(t) = F(t)$

We recall that with the standard change $X = Y_1, X^{(1)} = Y_2$, the Cauchy problem

$$X^{(2)}(t) - AX(t) = 0, \quad X(0) = C_0, \quad X^{(1)}(0) = C_1, \quad A, C_0, C_1 \in \mathbb{C}_{n\times n} \tag{3.1}$$

is equivalent to the first order linear system

$$Y^{(1)}(t) = C_L Y(t); \quad Y(0) = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}. \tag{3.2}$$

Thus problem (3.1) has only one solution $X(t)$ given by $X(t) = [I, 0] \exp\,(tC_L) \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$ (see [2], [6], p.122). The next lemma provides a representation for the general solution of the matrix equation (1.1) that is analogous to the well-known representation of the general solution of (1.1) for the scalar case.

Lemma 2. *Let us suppose that the matrix $A \in \mathbb{C}_{n\times n}$ has a fundamental pair of co-square roots $(X_i, T_i), i = 1, 2$. Then the unique solution of problem (3.1) takes the form*

$$X(t) = X_1 \exp(tT_1)C + X_2 \exp(tT_2)D, \tag{3.3}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = V^{-1} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \tag{3.4}$$

where V is given by (2.2).

Proof. If C, D are arbitrary matrices in $\mathbb{C}_{n\times n}$, then $X_1 \exp(tT_1)C$ and $X_2 \exp(tT_2)D$ are solutions of the differential equation (1.1) because (X_i, T_i), for $i = 1, 2$, are co-square roots of A. In order to find the unique solution of (3.1) that satisfies the Cauchy conditions of (3.1), the matrices C, D appearing in (3.3) must verify

$$C_0 = X(0) = X_1C + X_2D; \quad C_1 = X^{(1)}(0) = X_1T_1C + X_2T_2D \tag{3.5}$$

because from (3.3) we have $X^{(1)}(t) = X_1T_1 \exp(tT_1)C + X_2T_2 \exp(tT_2)D$ for all real numbers t. System (3.5) may be written in the form

$$V \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_1T_1 & X_2T_2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}.$$

As (X_i, T_i), for $i = 1, 2$, define a fundamental pair of co-square roots of A, the matrix V is invertible and the result is proved.

Now Lemma 2 suggests that in an analogous way to the scalar case we can obtain a variation of the parameter method for solving equation (1.2), as it happens for the scalar case.

Theorem 3. *Let us consider problem (1.2) where F is a continuous $\mathbb{C}_{n\times n}$ valued matrix function, and let us suppose that A has a fundamental pair of co-square roots $(X_i, T_i), i = 1, 2$. If V is the block matrix defined by (2.2) and $W = V^{-1} = (W_{ij})$, where $W_{ij} \in \mathbb{C}_{n\times n}$, for $1 \le i,\ j \le 2$, then the general solution of problem (1.2) is given by the expression*

$$X(t) = X_1 \exp(tT_1)C(t) + X_2 \exp(tT_2)D(t), \tag{3.6}$$

$$C(0) \in \mathbb{C}_{n\times n}, \quad C(t) = C(0) + \int_0^t \exp(-sT_1)W_{12}F(s)ds,$$

$$D(0) \in \mathbb{C}_{n\times n}, \quad D(t) = D(0) + \int_0^t \exp(-sT_2)W_{22}F(s)ds. \tag{3.7}$$

Proof. Note that with the standard change $X = Y_1, X^{(1)} = Y_2$, and considering the equivalent first order extended linear system, problem (1.1) has only one solution when the Cauchy conditions are prescribed ([6], p.122).

Let us suppose that $C(t), D(t)$ are matrix functions satisfying the system

$$\begin{bmatrix} X_1 \exp(tT_1) & X_2 \exp(tT_2) \\ X_1T_1 \exp(tT_1) & X_2T_2 \exp(tT_2) \end{bmatrix} \begin{bmatrix} C^{(1)}(t) \\ D^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{3.8}$$

that may be written as

$$V[\mathrm{Diag}(\exp(tT_1), \exp(tT_2))] \begin{bmatrix} C^{(1)}(t) \\ D^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{3.9}$$

where V is defined by (2.2). FOr these matrix functions $C(t), D(t)$, the function $X(t)$ defined by (3.6) satisfies

$$X^{(1)}(t) = X_1T_1 \exp(tT_1)C(t) + X_2T_2 \exp(tT_2)D(t), \tag{3.10}$$

because from (3.8) we have $X_1 \exp(tT_1)C^{(1)}(t) + X_2 \exp(tT_2)D^{(1)}(t) = 0$. By differentiation in (3.10) we have

$$X^{(2)}(t) = X_1T_1^2 \exp(tT_1)C(t) + X_2T_2^2 \exp(tT_2)D(t) + F(t), \tag{3.11}$$

because from (3.8) we get $X_1T_1 \exp(tT_1)C^{(1)}(t) + X_2T_2 \exp(tT_2)D^{(1)}(t) = F(t)$.

From (3.6) and (3.11) it follows that

$$\begin{aligned} X^{(2)}(t) - AX(t) = {} & (X_1T_1^2 - AX_1) \exp(tT_1)C(t) + (X_2T_2^2 - AX_2) \\ & \cdot \exp(tT_2)D(t) + F(t) = F(t) \end{aligned}$$

because $X_iT_i^2 - AX_i = 0$ for $i = 1, 2$. Thus $X(t)$ given by (3.6), where $C(t), D(t)$ are defined by (3.8), is a solution of problem (1.2) on any interval containing the origin and where F is continuous. From (3.9) we get

$$\begin{bmatrix} C(t) \\ D(t) \end{bmatrix} = \begin{bmatrix} C(0) \\ D(0) \end{bmatrix} + \int_0^t [\mathrm{Diag}(\exp(-sT_1), \exp(-sT_2))]V^{-1} \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds. \tag{3.12}$$

In order to determine matrices $C(0)$ and $D(0)$, note that taking $t = 0$ in (3.4) and (3.10), we have

$$X(0) = X_1C(0) + X_2D(0),$$

$$X^{(1)}(0) = X_1T_1C(0) + X_2T_2D(0)$$

or

$$\begin{bmatrix} X(0) \\ X^{(1)}(0) \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ -X_1T_1 & X_2T_2 \end{bmatrix} \begin{bmatrix} C(0) \\ D(0) \end{bmatrix} = V \begin{bmatrix} C(0) \\ D(0) \end{bmatrix}. \tag{3.13}$$

From (3.12) and (3.13) the result is proved.

Now we will use the representation (3.6)–(3.7) of the general solution of equation (1.2) to find existence and uniqueness conditions for solutions of problem (1.4) as well as an explicit expression of these solutions in terms of data and a fundamental pair of co-square roots of the matrix A.

For the sake of clarity, in the statement of the next result we introduce the following block matrix:

$$S = \begin{bmatrix} E_1X_1 + E_2X_1T_1 & E_1X_2 + E_2X_2T_2 \\ (F_1X_1 + F_2X_1T_1)\exp(aT_1) & (F_1X_2 + F_2X_2T_2)\exp(aT_2) \end{bmatrix}. \tag{3.14}$$

Theorem 4. *Let us suppose that $A \in \mathbb{C}_{n\times n}$ has a fundamental pair of co-square roots $\{(X_i, T_i), i = 1, 2\}$, and let $F(t)$ be a continuous function on the interval $[0, a]$. Let S be defined by (3.14) and let Q be the matrix*

$$Q = -(F_1X_1 + F_2X_1T_1)\int_0^a \exp((a-s)T_1)W_{12}F(s)ds$$
$$-(F_1X_2 + F_2X_2T_2)\int_0^a \exp((a-s)T_2)W_{22}F(s)ds \tag{3.15}$$

where $V^{-1} = W = (W_{ij})$.

(i) *If the matrix S is invertible, then problem (1.4) has only one solution $X(t)$ defined by (3.6)–(3.7), where $\begin{bmatrix} C(0) \\ D(0) \end{bmatrix} = S^{-1}\begin{bmatrix} O \\ Q \end{bmatrix}$.*

(ii) *If S is singular, then problem (1.4) is solvable, if and only if the following property is satisfied:*

$$SS^+\begin{bmatrix} O \\ Q \end{bmatrix} = \begin{bmatrix} O \\ Q \end{bmatrix}. \tag{3.16}$$

In this case the solution set of problem (1.4) is given by (3.6)–(3.7), where

$$\begin{bmatrix} C(0) \\ D(0) \end{bmatrix} = S^+\begin{bmatrix} O \\ Q \end{bmatrix} + (I_{2n} - S^+S)Y \tag{3.17}$$

and Y is an arbitrary matrix in $\mathbb{C}_{2n\times n}$.

Proof. From Theorem 3, the general solution of equation (1.2) is given by the expression $X(t)$ defined by (3.6)–(3.7), where $C(0)$ and $D(0)$ are arbitrary matrices in $\mathbb{C}_{n\times n}$. In order to find solutions of problem (1.4), we have to determine appropriate matrices $C(0)$ and $D(0)$, such that placed in (3.7), the corresponding function $X(t)$ defined by (3.6) satisfies the boundary value conditions of problem (1.4). By the expressions (3.6), (3.10) and the proof of Theorem 3, it occurs if the matrices $C = C(0)$ and $D = D(0)$ satisfy the following algebraic system:

$$E_1(X_1C + X_2D) + E_2(X_1T_1C + X_2T_2D) = 0,$$
$$F_1(X_1\exp(aT_1)C(a) + X_2\exp(aT_2)D(a)) + F_2(X_1T_1\exp(aT_1)C(a) \tag{3.18}$$
$$+X_2T_2\exp(aT_2)D(a)) = 0.$$

Now, taking into account that $C(a) = C(0) + \int_0^a \exp(-sT_1)W_{12}F(s)ds$, and $D(a) = D(0) + \int_0^a \exp(-sT_2)W_{22}F(s)ds$, from (3.15) and (3.18), we get that matrices $C = C(0)$ and $D = D(0)$ must verify the algebraic system

$$S \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} O \\ Q \end{bmatrix}. \tag{3.19}$$

It is well known[10] that system (3.19) is solvable, if and only if the condition (3.16) is satisfied, and that in this case the solution set of system (3.19) is given by the expression (3.17). If S is invertible, then the only solution is $S^{-1} \begin{bmatrix} O \\ Q \end{bmatrix}$. Hence the result is established.

Remark 2. In order to characterize the existence of solutions for the boundary value problem (1.4), we have to consider whether the condition (3.16) is verified and we need to compute S^+. An easy method for computing S^+ may be found in [3, p.12]. In the next example we show a problem of the type (1.4), where it is not possible to find a pair of square roots of the matrix A satisfying the conditions required in [9], and for which the method presented here provides the solution of the problem.

Example 4. Let us consider the problem (1.4) in $\mathbb{C}_{2\times 2}$ with the following data:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \; E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \; E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; F_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}; \; F_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix};$$

$F(t)$ continuous, $a > 0$. It is easy to show that matrix A has only two square roots given by $A_1 = 2^{-1/2}A$, $A_2 = -2^{-1/2}A$, and as A is singular, the difference $A_1 - A_2 = 2^{1/2}A$ is singular. Note that $\sigma(A) = \{0, 2\}$, and $\sigma(C_L) = \{0, 2^{1/2}, -2^{1/2}\}$. Thus the Jordan canonical form of A takes the form $J_A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. From Theorem 2, the Jordan canonical form of C_L is given by

$$J_{C_L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{1/2} & 0 \\ 0 & 0 & 0 & -2^{1/2} \end{bmatrix} \quad \text{and} \quad J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 2^{1/2} & 0 \\ 0 & -2^{1/2} \end{bmatrix}.$$

Also from Theorem 1, there exists a fundamental pair of co-square roots of A, defined by (P_{11}, J_1), (P_{12}, J_2), where $P_{11}J_1^2 = AP_{11}$ and $P_{12}J_2^2 = AP_{12}$. As $J_1^2 = 0$, this system is equivalent to solving

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} P_{11} = 0 \quad \text{and} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} P_{12} = 0.$$

The set of all nonzero solutions of these equations is given by

$$P_{11} = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}, \quad |a|^2 + |b|^2 > 0 \text{ and } P_{12} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}, \quad |c|^2 + |d|^2 > 0.$$

Taking $a = c = d = 1, b = 0$, we get $P_{11} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, P_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and the corresponding matrix V defined by (2.2) is invertible with

$$V = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 2^{1/2} & -2^{1/2} \\ 0 & -1 & 2^{1/2} & -2^{1/2} \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & (2^{1/2}4)^{-1} & (2^{1/2}4)^{-1} \\ 1/4 & 1/4 & -(2^{1/2}4)^{-1} & -(2^{1/2}4)^{-1} \end{bmatrix}.$$

In our case the matrix S defined by (3.14) takes the form

$$S = \begin{bmatrix} 0 & 1 & 2 + 2^{1/2} & 2 - 2^{1/2} \\ 1 & -1 & 1 + 2^{1/2} & 1 - 2^{1/2} \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 + 2^{3/2}\exp(a2^{1/2}) & -2 - 2^{3/2}\exp(-2^{1/2}a) \end{bmatrix}.$$

As S is invertible, from Theorem 4 – (i), for any continuous function $F(t)$, problem (1.4) with the above data has only one solution given by the corresponding expressions (3.6)–(3.7).

References

[1] T.M. Apostol, Explicit formulas for solutions of the second order matrix differential equation $Y'' = AY$, *Amer. Math. Monthly,* **82**:2 (1975), 159–162.

[2] R.W. Brockett, Finite Dimensional Linear Systems, John Wiley and Sons, New York, 1970.

[3] S.L. Campbell and C.D. Meyer, Jr., Generalized Inverses of Linear Transformations, Pitman, San Francisco, 1979.

[4] G.W. Cross and P. Lancaster, Square roots of complex matrices, *Linear and Mult. Algebra,* **1** (1974), 280–293.

[5] I. Gohberg, P. Lancaster and L. Rodman, Matrix Polynomials, Academic Press, 1982.

[6] T.M. Flett, Differential Analysis, Cambridge Univ. Press, 1980.

[7] P. Lancaster, Lambda Matrices and Vibrxating Systems, Pergamon, Oxford, 1966.

[8] P. Lancaster and M. Tismenetsky, The Theory of Matrices, Sec. Ed., Academic Press, 1985.

[9] L. Jódar, Explicit expressions for Sturm-Liouville operator problems, Proc. of the Edinburgh Math. Soc., 1987, 30, 301–309.

[10] C.R. Rao and S.K. Mitra, Generalized Inverses of Matrices and Its Applications, John Wiley, New York, 1971.