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A POSTERIORI ERROR ANALYSIS FOR MIXED FINITE ELEMENT SOLUTION OF THE TWO-DIMENSIONAL STATIONARY STOKES PROBLEM*

Yuan Wei
(*Department of Applied Mathematics, Tsinghua University, Beijing, China*)

Abstract

In this paper we present a posteriori error estimator in a suitable norm of mixed finite element solution for the two-dimensional stationary Stokes problem. The estimator is optimal in the sense that, up to multiplicative constants, the upper and lower bounds of the error are the same. The constants are independent of the mesh and the true solution of the problem.

§1. Introduction

The stationary Stokes problem arises from stationary flow of an incompressible viscous fluid with a small Renord's number. However, it is the basis of handling the complete Navier-Stokes equations. In this paper we consider solving the Stokes problem by the mixed finite element method. It is difficult to solve the Stokes problems with singularities, including, for example, corner singularities. The adaptive finite element method is, however, a class of effective method for solving the boundary value problems with singularities. A posteriori error estimator of finite element methods presents an appreciation for the computed results. It is the basis of the adaptive refinement mesh algorithm.

So far, the theorecal problems of a posteriori error analysis for finite element methods of the boundary value problems of one-dimensional elliptic equations have been solved by I. Babuska and others (see [9]–[12]). For two-dimensional problems there are also a large number of works by I. Babuska and his co-workers. They have presented a posteriori error indicator of the finite element method for the Dirichlet problem of Poisson equation ([1]–[5]). The indicator in [3], for example, is based on solving local Dirichlet problems in the patch of elements surrounding each vertex in the finite element mesh. And in [1], using conforming bilinear square elements, I. Babuska and A. Miller show that error indicators can be based on jumps in the normal derivative of the computed solution at interelement boundaries. Such schemes as a rule require less computation than the ones involving the solution of local Dirichlet problems. E Wei-nan and Huang Hong-ci extended some results of I. Babuska to the case of more general conforming elements (see [7] and [12]). In [6] R. E. Bank and A. Weiser present error indicators for the Neumann problem of an elliptic equation by solving a local Neumann problem in each finite element.

In this paper, a posteriori error estimator of mixed finite element solution for the two-dimensional stationary Stokes problem is presented in a suitable norm. The estimator is optimal in the sense that, up to multiplicative constants, the upper and lower bounds of the error are the same. The constants are independent of the mesh and the true solution of the Stokes problem. Moreover, they are not large in practice. The estimator is based on solving local Stokes problems in the patch of elements surrounding each vertex in the finite

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element mesh. In this scheme, the Dirichlet boundary conditions ensure well-posedness of the local Stokes problems. The estimator we obtain finally consists of the indicator at each element and those indicators which are based on the computation of the norm of the local residual of the Stokes equation and the jumps in the computed pressure solution and in the normal derivative of the computed velocity solution at interelement boundaries. Two numerical examples in this paper support the above theorecal results.

In Section 2, we shall introduce the results of the mathematical theory and the mixed finite element methods of the Stokes problem, explain the basic notation used in this paper and present a partition of the solution domain, and give an approximate computing method for the LBB constant of each element. In Section 3, the main theorems in the paper will be proved and a posteriori error estimator will be presented. Finally, two numerical examples in Section 4 support the results in Section 3.

§2. Preliminaries

Let Ω be a bounded connected domain in R^2 with a smooth boundary Γ. We consider the Dirichlet problem of Stokes equation

$$\begin{cases} -\nu\cdot\nabla^2\vec{u}+\vec{\text{grad}}\cdot p=\vec{f} & \text{in } \Omega, \\ \text{div}\cdot\vec{u}=g & \text{in } \Omega, \\ \vec{u}=0 & \text{on } \Gamma \end{cases} \tag{2.1}$$

where the velocity function $\vec{u}(x)$ and the pressure function $p(x)$ are unknown, ν is a viscous coefficient, and $\vec{f}(x)$ and $g(x)$ are the given functions. For simplicity, we assume that $\nu=1$.

In this paper, $H^m(\Omega)$ with m being an integer denotes the usual Sobolev space. $H_0^1(\Omega)$ denotes the space in which the functions are in $H^1(\Omega)$ and their traces are zero. The norm of $H_0^1(\Omega)$ is defined as

$$|u|_{1,\Omega}=\left\{\iint_\Omega[(\partial u/\partial x_1)^2+(\partial u/\partial x_2)^2]dx\right\}^{1/2}, \qquad u\in H_0^1(\Omega).$$

Correspondingly, the norm of $(H_0^1(\Omega))^2$ is defined as

$$|\vec{u}|_{1,\Omega}=[|u_1|_{1,\Omega}^2+|u|_{1,\Omega}^2]^{1/2}, \qquad \vec{u}=(u_1,u_2)\in(H_0^1(\Omega))^2.$$

Denote $V(H_0^1(\Omega))^2$ and $W=L_0^2(\Omega)=\{q\in L^2(\Omega);\ \iint_\Omega q(x)dx=0\}$.

The variational form of (2.1) is

$$\begin{cases} \text{Find } [\vec{u},p]\in V\times W, \text{ such that} & \\ \quad a(\vec{u},\vec{v})+b(\vec{v},p)=(\vec{f},\vec{v}) & \text{for all } \vec{v}\in V, \\ \quad b(\vec{u},q)=(-g,q) & \text{for all } q\in W \end{cases} \tag{2.2}$$

where

$$a(\vec{u},\vec{v})=\iint_\Omega(\vec{\text{grad}}u_1\cdot\vec{\text{grad}}\ v_1+\vec{\text{grad}}\ u_2\cdot\vec{\text{grad}}\ v_2)dx,$$

$$b(\vec{u},q)=-\iint_\Omega \text{div}\cdot\vec{u}(x)q(x)dx,$$

$$(\vec{f},\vec{v})=\iint_\Omega\vec{f}\cdot\vec{v}dx \qquad \text{and} \qquad (g,q)=\iint_\Omega g\cdot q dx.$$

It is obvious that there exists a constant $\beta > 0$ depending only on Ω such that

$$\inf_{q\in W} \sup_{\vec{v}\in V} \frac{|b(\vec{v},q)|}{|\vec{v}|_{1,\Omega}\cdot \|q\|_{0,\Omega}} = \beta$$

where the constant β is called LBB constant of Ω (see [8]). Moreover, we can easily prove the following lemma.

Lemma 2.1. *For any $q \in L_0^2(\Omega)$, the following inequality holds:*

$$\sup_{\vec{v}\in V} \frac{|b(\vec{v},q)|}{|\vec{v}|_{1,\Omega}} \leq \|q\|_{o,\Omega}.$$

Therefore, $0 < \beta < 1$ and $|||q||| = \sup_{\vec{v}\in V} \frac{|b(\vec{v},q)|}{|\vec{v}|_{1,\Omega}}$ is a norm in W which is equivalent to the norm $\|q\|_{1,\Omega}$ for $q \in W$.

One can prove that, if $f \in (H^{-1}(\Omega))^2$ and $g \in L_0^2(\Omega)$, then there exists a unique solution of (2.2), $[\vec{u},p] \in V \times W$. Moreover, when the boundary is sufficiently smooth, and if $\vec{f} \in (L^2(\Omega))^2$ and $g \in H^1(\Omega) \cap L_0^2(\Omega)$, then the unique solution of (2.2), $[\vec{u},p]$, is in $(H^2(\Omega))^2 \times H^1(\Omega)$ and there is a constant $C > 0$, such that

$$\|\vec{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \leq C\cdot(\|\vec{f}\|_{0,\Omega} + \|g\|_{1,\Omega})$$

where the constant C is independent of the solution $[\vec{u},p]$ (see [8]).

Now we give a partition $\Delta = \{k_j\}_{j=1}^m$ of the domain Ω. The partition is either rectangular or triangulr. And we assume that Δ is a quasiuniform regular partition, i.e., the following conditions are satisfied:

a) There exists a constant $\sigma > 0$ such that $0 < \sigma < h_j/\rho_j$, for any $k_j \in \Delta$;

b) There exists a constant $\gamma > 0$ such the $\gamma < h_j/h_\Delta \leq 1$, for any $k_j \in \Delta$

where h_j denotes the diameter of k_j, ρ_i that of the maximum circle contained in k_j, and $h_\Delta = \max_j\{h_j\}$.

For the rectangular partition, set

$$S^h = \{\varphi \in C(\Omega);\ \ \varphi|_{k_j} \in Q_2(k_j),\ \ j = 1,\cdots,m\};$$

and for the triangular partition, set

$$S^h = \{\varphi \in C(\Omega);\ \ \varphi|_{k_j} \in P_2(k_j),\ \ j = 1,\cdots,m\},$$

where $Q_2(k_j)$ denotes the space spanned by the bi-quadratic polynomials on k_j, and $P_2(k_j)$ the space spanned by the quadratic polynomials on k_j. We take

$$V_h = (S^h \cap H_0^1(\Omega))^2 \tag{2.5}$$

and

$$W_h = \{q \in L_0^2(\Omega);\ \ q|_{k_j} = \text{ const.},\ \ j = 1,\cdots,m\}.$$

Obviously, $V_h \subset V$ and $W_h \subset W$. A mixed finite element approximation can be written as

$$\begin{cases} \text{Find } [\vec{u}_h,p_h] \in V_h \times W_h, \text{ such that} & \\ \qquad a(\vec{u},\vec{v}) + b(\vec{v},p) = (\vec{f},\vec{v}) & \text{for all } \vec{v} \in V_h, \\ \qquad b(\vec{u},q) = (-g,q) & \text{for all } q \in W_h. \end{cases} \tag{2.6}$$

It can be proved that (2.6) has a unique solution, and an asymptotic error estimate is as follows:

$$|\vec{u}-\vec{u}_h|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\le C\cdot\{\inf_{\vec{v}\in V_h}|\vec{u}-\vec{v}|_{1,\Omega}+\inf_{q\in W_h}\|p-q\|_{0,\Omega}\},$$

where the constant C is independent of the solution $[\vec{u},\vec{p}]$ and Δ (see [8]).

In order to obtain a posteriori estimator of mixed finite solution $[\vec{u}_h,p_h]$, we consider a partition of unity

$$\Phi=\{\phi_1,\cdots,\phi_M\},\quad \phi_i\in H^1(\Omega)\ ,i=1,\cdots,M;\ \ \phi_i(x)\ge 0,\ \ \sum_{i=1}^{M}\phi_i(x)=1,\ \ x\in\Omega \quad (2.7)$$

of the domain Ω, Supp ϕ_i and Supp ${}^0\phi_i$ denote the support of ϕ_i and its interior, respectively. It is always possible to partition the set such that

$$\Phi=\bigcup_{l=1}^{r(\Phi)}\Phi_l,\quad \Phi_l\cap\Phi_j=\emptyset\quad \text{for } l\ne j \tag{2.8a}$$

and that the interiors of the supports of the members of each Φ_l are disjoint, that is

$$\phi_i,\phi_j\in\Phi_l\quad\text{and}\quad i\ne j\Rightarrow\quad \text{supp }{}^0\phi_i\cap\text{ Supp }{}^0\Phi_j=\emptyset. \tag{2.8b}$$

For instance, it suffices to let each Φ_l consist of exactly one. The smallest integer r for which (2.8) holds is the overlap index $r(\Phi)$ of Φ.

For given $\Phi=\{\phi_i\}_{i=1}^{M}$ and $\Delta=\{k_j\}_{j=1}^{m}$, we define the index set

$$\chi_j=\chi_j(\Phi,\Delta)=\{i=\{1,\cdots,M\};k_j\cap\text{ supp }{}^0\phi_i\ne\emptyset\},\ \ j=1,\cdots,m. \tag{2.9}$$

Then (2.8) implies that

$$\sum_{i\in\chi_j}\phi_i(x)=1,\quad \text{for any } x\in k_j,\ \ j=1,\cdots,m. \tag{2.10}$$

The maximum cardinality $\max\{|\chi_j|;j=1,\cdots,m\}$ will be said to be in intersection index $\chi(\Phi,\Delta)$ of Φ and Δ.

We consider a family $\mathcal{F}$ of triples $\{\Phi,\Delta,V_h\}$ each of which consists of a partition of unity Φ (cf. 2.7)), a set partition Δ and a finite element subspace V_h of V. The family $\mathcal{F}$ will be said to be admissible if it satisfies the following four conditions:

1°. There exists a constant $r(\mathcal{F})>0$ depending only on $\mathcal{F}$ such that

$$r(\Phi)\le r(\mathcal{F}),\qquad \forall(\Phi,\Delta,V_h)\in\mathcal{F}.$$

2°. There exists a constant $\chi(\mathcal{F})>0$ depending only on $\mathcal{F}$ such that

$$\chi(\Phi,\Delta)\le\chi(\mathcal{F}),\qquad \forall(\Phi,\Delta,V_h)\in\mathcal{F}.$$

3°. There exists a constant $C_0(\mathcal{F})>0$ depending only on $\mathcal{F}$ such that

$$|D^\alpha\phi_i(x)|\le C_0h_j^{-|\alpha|},\forall x\in\Omega,\ \ i\in\chi_j(\Phi,\Delta),0\le|\alpha|\le 1,\ \ \forall(\Phi,\Delta,V_h)\in\mathcal{F}$$

where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $\alpha_i \geq 0$ integer.

4°. There exists a constant $C_1(\mathcal{F}) > 0$ depending only on $\mathcal{F}$ such that for any $(\Phi, \Delta, V_h) \in \mathcal{F}$ and any $\vec{v} \in V$, we can find a function $\vec{\psi} \in V_h$ for which

$$\|\vec{v} - \vec{\psi}\|_{s,k_1} \leq C_1 h^{t-s} |\vec{v}|_{t,k}, \quad \forall k_1 \in \Delta, \quad 0 \leq s \leq 1$$

where k is a standard reference element.

The following lemma, which was proved in [3], will play an essential role in the further study.

Lemma 2.2. *Let $\mathcal{F}$ be an admissible family of triples (Φ, Δ, V_h). Then there exists a constant $C(\mathcal{F}) < \infty$ depending only on $\mathcal{F}$ such that for any (Φ, Δ, V_h),*

$$\inf_{\vec{\psi} \in V_h} \sum_{i=1}^{M} |\phi_i \cdot (\vec{v} - \vec{\psi})|_{1,\Omega}^2 \leq C(\mathcal{F})^2 |\vec{v}|_{1,\Omega}^2.$$

In order to obtain a posteriori error estimator of $[\vec{u}_h, p_h]$, we also need to estimate the LBB constant β_j of each element $k_j, j = 1, \cdots, m$, where

$$\beta_j = \inf_{q \in L_0^2(k_j)} \sup_{\vec{v} \in (H_0^1(k_j))^2} \frac{|b_{k_j}(\vec{v}, q)|}{|\vec{v}|_{1,k_j} \|q\|_{0,k_j}}$$

where

$$b_{k_j}(\vec{v}, q) = -\iint_{k_j} \mathrm{div} \cdot \vec{v} q dx.$$

It is easy to see that β_j is an invarious value for rotation and reflection transformation. Therefore we have

Theorem 2.3. *The LBB constant of k_j depends only on the shape of k_j, but β_j is independent of the size of k_j.*

Let k be a reference element of the finite element mesh $\Delta = \{k_j\}_{j=1}^m$. For the rectangular partition, k is a square, and for the triangular one k is a standard equicrural and right triangle. Set

$$\beta = \inf_{q \in L_0^2(k)} \sup_{\vec{v} \in (H_0^1(k))^2} \frac{|b_k(\vec{v}, q)|}{|\vec{v}|_{1,k} \|q\|_{0,k}}.$$

Since the partition Δ is regular, and for all $k_j \in \Delta$ there are affine mappings $F_j, j = 1, \cdots, m$, such that $F_j(k) = k_j$, associating with Theorem 2.3, it is not difficult to prove

Theorem 2.4. *For the above partition Δ, there exists a constant $\tilde{\sigma} > 1$, such that for all k_j*

$$\beta_j \geq \beta(k)/\tilde{\sigma} \tag{2.12}$$

where the constant $\tilde{\sigma}$ depends only on the regularity coefficient σ. In particular, $\tilde{\sigma} = \sigma$ for the rectanglar partition.

Since (2.12) is a saddle point problem, one can compute approximately the LBB constant $\beta(k)$ of the reference element k by some numerical methods.

§3. The Main Results

Let $\vec{e} = \vec{u} - \vec{u}_h$ and $\epsilon = p - p_h$. Then $\vec{e}$ and ε satisfy the following error equation

$$\begin{cases} a(\vec{e}, \vec{v}) + b(\vec{v}, \varepsilon) = 0, & \forall \vec{v} \in V_h, \\ b(\vec{e}, q) = 0, & \forall q \in W_h. \end{cases} \tag{3.1}$$

Throughout this section, we suppose that the family $\mathcal{F}$ of triples $\{\Phi, \Delta, V_h\}$ is admissible. Now we estimate the error $|\vec{e}|_{1,\Omega} + |||\varepsilon|||$.

Lemma 3.1. *For any function $\vec{v}$ in V, we have*

$$\inf_{\vec{v}_h \in V_h} |a(\vec{e}, \vec{v} - \vec{v}_h) + b(\vec{v} - \vec{v}_h, \varepsilon)| \le C(\mathcal{F}) \cdot \hat{\eta} \cdot |\vec{v}|_{1,\Omega}, \tag{3.2}$$

where the constant $C(\mathcal{F})$ is the same as that in Lemma 2.2, *and*

$$\hat{\eta} = (\sum_{i=1}^{M} \hat{\eta}_i^2)^{1/2}, \quad \hat{\eta}_i = \sup_{\vec{v} \in V} \frac{|a(\vec{e}, \phi_i \cdot \vec{v}) + b(\phi_i \cdot \vec{v}, \varepsilon)|}{|\phi_i \cdot \vec{v}|_{1,\Omega}}. \tag{3.3}$$

Proof. For any given function $\vec{v}$ in V, since Φ is a partition of the unity of Ω, we have

$$\begin{aligned} a(\vec{e}, \vec{v} - \vec{v}_h) + b(\vec{v} - \vec{v}_h, \varepsilon) &= \Big| \sum_{i=1}^{M(\Phi)} \Big[a(\vec{e}, \phi_i \cdot (\vec{v} - \vec{v}_h)) + b(\phi_i \cdot (\vec{v} - \vec{v}_h), \varepsilon) \Big] \Big| \\ &\le \sum_{i=1}^{M} \hat{\eta} \cdot |\phi_i \cdot (\vec{v} - \vec{v}_h)|_{1,\Omega}. \end{aligned}$$

Thus, from Lemma 2.2 and the Schwarz inequality,

$$\begin{aligned} \inf_{\vec{v}_h \in V_h} |a(\vec{e}, \vec{v} - \vec{v}_h) + b(\vec{v} - \vec{v}_h, \varepsilon)| &\le \Big(\sum_{i=1}^{M} \hat{\eta}_i^2 \Big)^{1/2} \inf_{\vec{v}_h \in V_h} \Big(\sum_{i=1}^{M} |\phi_i \cdot (\vec{v} - \vec{v}_h)|_{1,\Omega}^2 \Big)^{1/2} \\ &\le C(\mathcal{F}) \cdot \hat{\eta} \cdot |\vec{v}|_{1,\Omega}. \end{aligned}$$

Lemma 3.2. *There exists a function $q_h \in W_h$ such that*

$$\|\varepsilon - q_h\|_{0,\Omega} \le \beta^{-1} \cdot \tilde{\sigma} \cdot (\tilde{\eta} + |\vec{e}|_{1,\Omega}) \tag{3.4}$$

where β and $\tilde{\sigma}$ are defined in Section 2, and

$$\tilde{\eta} = \Big(\sum_{j=1}^{m} \tilde{\eta}_j^2 \Big)^{1/2}, \quad \tilde{\eta}_j = \sup_{\substack{\vec{v} \in V \\ \mathrm{Supp}^0 \vec{v} \subset k_j}} \frac{|a(\vec{e}, \vec{v}) + b(\vec{v}, \varepsilon)|}{|\vec{v}|_{1,\Omega}}. \tag{3.5}$$

proof. Let $q_h|_{k_j} = \dfrac{1}{\mathrm{meas}(k_j)} \iint_{k_j} \varepsilon dx$. Thus q_h is a constant in each element and

$$\iint_\Omega q_h dx = \iint_\Omega \varepsilon dx = \iint_\Omega (p - p_h) dx = 0.$$

Therefore $q_h \in W_h$, and

$$(\varepsilon - q_h)|_{k_j} \in L_0^2(k_j), \quad (\varepsilon - q_h, q_h)_{0,k_j} = \iint_{k_j} (\varepsilon - q_h) \cdot q_h dx = 0.$$

Thus , from the LBB condition, for k_j we have

$$\|\varepsilon - q_h\|_{0,k_j} \le \beta_j^{-1} \sup_{\vec{v} \in (H_0^1(k_j))^2} \frac{|b_{k_j}(\vec{v}, \varepsilon - q_h)|}{|\vec{v}|_{1,k_j}}. \tag{3.6}$$

For any $\vec{v} \in (H_0^1(k_j))^2$, since

$$-b_{k_j}(\vec{v}, q_h) = -\iint_{k_j} \operatorname{div} \cdot \vec{v} q dx = -q_h|_{k_j} \cdot \iint_{k_j} \operatorname{div} \cdot \vec{v} dx = 0,$$

(3.6) implies

$$\|\varepsilon - q_h\|_{0,k_j} \le \beta_j^{-1} \sup_{\vec{v} \in (H_0^1(k_j))^2} \frac{|b_{k_j}(\vec{v}, \varepsilon)|}{|\vec{v}|_{1,k_j}}.$$

On the other hand, for any $\vec{v} \in V$ satisfying $\mathrm{Supp}^0 \vec{v} \subset k_j$, from (3.1), we have

$$|b_{k_j}(\vec{v}, \varepsilon)| \quad = |b(\vec{v}, \varepsilon)| \le |a(\vec{e}, \vec{v}) + b(\vec{v}, \varepsilon)| + |a(\vec{e}, \vec{v})| \le (\eta_j + |\vec{e}|_{1,k_j}) \cdot |\vec{v}|_{1,\Omega}.$$

Therefore we have

$$\|\varepsilon - q_h\|_{0,k_j} \le (\eta_j + |\vec{e}|_{1,k_j})/\beta_j,$$

and

$$\begin{aligned}\|\varepsilon - q_h\|_{0,\Omega} \quad &\le \Big[\sum_{j=1}^m (\tilde{\eta}_j + |\vec{e}|_{1,k_j})^2/\beta_j^2\Big]^{1/2} \\ &\le \Big[\sum_{j=1}^m (\tilde{\eta}_j + |\vec{e}|_{1,k_j})^2\Big]^{1/2} \tilde{\sigma}/\beta \le (\tilde{\eta} + |\vec{e}|_{1,\Omega}) \cdot \tilde{\sigma}/\beta,\end{aligned}$$

where the constants $\tilde{\sigma}$ and β are defined in Section 2.

Lemma 3.3. *Let α, β and x be nonnegative real numbers. If x satisfies*

$$x^2 \le (\alpha + \beta) \cdot x + \alpha \cdot \beta,$$

then

$$x \le (1 + \sqrt{2}) \cdot (\alpha + \beta)/2.$$

The proof this Lemma is very easy.

Theorem 3.1. *Under the above assumptions, we have*

$$|\vec{e}|_{1,\Omega} + |||\varepsilon||| \le C(\mathcal{T}) \cdot (2 + \sqrt{2}) \cdot \eta + \tilde{\sigma} \cdot (1 + \sqrt{2}) \cdot \varsigma/\beta \tag{3.7}$$

where $\eta = \max\{\tilde{\eta}, \hat{\eta}\}$ *and* $\varsigma = \|g - \operatorname{div} \cdot \vec{u}_h\|_{0,\Omega}$.

Proof. From the error equation (3.1), for any $\vec{v}_h \in V_h$,

$$|\vec{e}|^2 = |a(\vec{e}, \vec{e})| = |a(\vec{e}, \vec{e} - \vec{v}_h) + b(\vec{e} - \vec{v}_h, \varepsilon)| + |b(\vec{e}, \varepsilon)|.$$

Thus

$$|\vec{e}|_{1,\Omega}^2 \le \inf_{\vec{v}_h\in V_h} |a(\vec{e},\vec{e}-\vec{v}_h)+b(\vec{e}-\vec{v}_h,\varepsilon)|+|b(\vec{e},\varepsilon)|$$

$$\le C(\mathcal{T})\cdot\eta\cdot|\vec{e}|_{1,\Omega}+|b(\vec{e},\varepsilon)|.$$

For the second term on the right-hand side of the above inequality, if we take q_h as that in Lemma 3.2, then

$$|b(\vec{e},\varepsilon)| = |b(\vec{u}-\vec{u}_h,\varepsilon-q_h)| = |(g-\mathrm{div}\cdot\vec{u}_h,\varepsilon-q_h)|$$

$$\le \|g-\ \mathrm{div}\cdot\vec{u}_h\|_{0,\Omega}\cdot\|\varepsilon-q_h\|_{0,\Omega}\le\tilde{\sigma}\cdot\varsigma\cdot(\eta+|\vec{e}|_{1,\Omega})/\beta.$$

Thus, we have

$$|\vec{e}|_{1,\Omega}^2\le(C(\mathcal{T})\cdot\eta+\tilde{\sigma}\cdot\varsigma/\beta)\cdot|\vec{e}|_{1,\Omega}+\tilde{\sigma}\cdot\varsigma\cdot\eta/\beta.$$

From Lemma 3.3,

$$|\vec{e}|_{1,\Omega}\le(1+\sqrt{2})\cdot(C(\mathcal{T})\cdot\eta+\tilde{\sigma}/\beta).$$

On the other hand,

$$|||\varepsilon||| = \sup_{\vec{v}\in V}\frac{|b(\vec{v},\varepsilon)|}{|\vec{v}|_{1,\Omega}}\le\sup_{\vec{v}\in V}\frac{|a(\vec{e},\vec{v}-\vec{v}_h)+b(\vec{v}-\vec{v}_h),\varepsilon)|}{|\vec{v}|_{1,\Omega}|}+\sup_{\vec{v}\in V}\frac{|a(\vec{e},\vec{v})|}{|\vec{v}|_{1,\Omega}}$$

$$\le[1+(1+\sqrt{2})/2]\cdot C(\mathcal{T})\cdot\eta+(1+\sqrt{2})\cdot\tilde{\sigma}\cdot\varsigma/(2\beta).$$

Therefore.

$$|\vec{e}|_{1,\Omega}+|||\varepsilon|||\le C(\mathcal{T})(2+\sqrt{2})\cdot\eta+(1+\sqrt{2})\cdot\tilde{\sigma}\cdot\varsigma/\beta.$$

So far, we have given an upper bound of $|\vec{e}|_{1,\Omega}+|||\varepsilon|||$. Now we present a lower bound of the error norm $|\vec{e}|_{1,\Omega}+\|\varepsilon\|_{0,\Omega}$ instead of $|\vec{e}|_{1,\Omega}+|||\varepsilon|||$.

Lemma 3.4. *Let X and Y be* Hilbert *spaces and X be a subspace of Y. $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm in Y, respectively. Then, for any given $y\in Y$, we have*

$$\sup_{x\in X}\frac{|(x,y)|}{\|x\|}=\|z\| \tag{3.9}$$

where $z\in X$ is the projection of y from Y into X.

Lemma 3.5. *Let Y be a* Hilbert *space, and X_i be a subspace of $Y, i=1,\cdots,n$. Assume that any two subspaces X_i and $X_j (i\ne j)$ are orthogonal to each other, $Y=\bigotimes_{i=1}^{n}X_i$. For any $x\in Y$,*

$$x=\sum_{i=1}^{n}x_i\in\tilde{Y}\subset Y,\qquad x_i\in X_i. \tag{3.10}$$

Then

$$\sup_{x\in Y}\frac{|(x,y)|}{\|x\|}=\Big\{\sum_{i=1}^{n}\Big[\sup_{x_i\in X_i}\frac{|(y,x_i)|}{\|x_i\|}\Big]^2\Big\}^{1/2}.$$

Proof. Since any two subspaces X_i and $X_j (i\ne j)$ are orthogonal to each other, therefore

$$\|x\|^2=\Big(\sum_{i=1}^{n}x_i,\sum_{j=1}^{n}x_j\Big)=\sum_{i=1}^{n}\|x_i\|^2.$$

Let z_i be the projection of $y \in Y$ into X_i. Then from Lemma 3.4,

$$\|z_i\| = \sup_{x_i \in X_i} \frac{|(y, x_i)|}{\|x_i\|}.$$

Hence

$$\sup_{x \in Y} \frac{|(y, x_i)|}{\|x\|} = \sup_{x \in Y} \frac{\left|\sum_{i=1}^{n}(y, x_i)\right|}{\|x\|} = \sup_{x \in Y} \frac{\left|(\sum_{i=1}^{n} z_i, \sum_{j=1}^{n} x_j)\right|}{\|x\|}$$

$$= \|\sum_{i=1}^{n} z_i\| = \Big[\sum_{i=1}^{n} \|z_i\|^2\Big]^{1/2}.$$

Theorem 3.2. *For the error $\vec{e}$ and ε we have following estimate of the lower bound*

$$\|\vec{e}\|_{1,\Omega} + \|\varepsilon\|_{0,\Omega} \geq (\hat{\eta}/\sqrt{r(\mathcal{F})} + \varsigma)/2. \tag{3.11}$$

Proof. Set

$$H_l = \{\vec{w} = \sum_{\phi_i \in \Phi_l} \phi_i \cdot \vec{v}; \vec{v} \in V\} \subset (H_0^1(\Omega))^2 \quad \text{and} \quad L_l = \{\operatorname{div} \cdot \vec{w}; \vec{w} \in H_l) \subset L_0^2(\Omega).$$

From (2.8b), for any ϕ_i and ϕ_j in $\Phi_l(i \neq j), a(\phi_i \cdot \vec{v}, \phi_j \cdot \vec{v}) = 0$. Thus H_l can be decomposed into the direct sum of many subspaces of which any two ones are orthogonal to each other:

$$H_l = \bigotimes_i H_l^{(i)} \quad \text{with} \quad \hat{H}_l^{(i)} = \{\vec{w} = \phi_i \cdot \vec{v}; \phi_i \in \Phi_l, \vec{v} \in V\}.$$

Let

$$\hat{\eta}_i^{(1)} = \sup_{\vec{v} \in V} \frac{a(\vec{e}, \phi_i \vec{v})}{|\phi_i \cdot \vec{v}|_{1,\Omega}} \quad \text{and} \quad \hat{\eta}_i^{(2)} = \sup_{\vec{v} \in V} \frac{|b(\phi_i \cdot \vec{v}, \varepsilon)|}{|\phi \cdot \vec{v}|_{1,\Omega}}.$$

Thus from Lemma 3.5 we obtain

$$\sup_{\vec{v} \in V} \frac{a(\vec{e}, \vec{v})}{|\vec{v}|_{1,\Omega}} \geq \sup_{\vec{w} \in H_l} \frac{a(\vec{e}, \vec{w})}{|\vec{w}|_{1,\Omega}} = \Big[\sum_{\phi_i \in \Phi_l} (\hat{\eta}_i^{(1)})^2\Big]^{1/2}.$$

it implies

$$|\vec{e}|_{1,\Omega} = \sup_{\vec{v} \in V} \frac{a(\vec{e}, \vec{v})}{|\vec{v}|_{1,\Omega}} \geq \Big[\sum_{i=1}^{M} (\hat{\eta}_i^{(1)})^2\Big]^{1/2} / \sqrt{r(\mathcal{F})}.$$

Along the line of the above proof for H_l, we can prove

$$\|\varepsilon\| = \sup_{q \in W} \frac{|(q, \varepsilon)|}{\|q\|_{0,\Omega}} \geq \Big[\sum_{i=1}^{M} (\hat{\eta}_i^{(2)})^2\Big]^{1/2} / \sqrt{r(\mathcal{F})}.$$

Hence

$$|\vec{e}|_{1,\Omega} + \|\varepsilon\|_{0,\Omega} \geq \Big[\sum_{i=1}^{M} (\hat{\eta}_i^{(1)} + \hat{\eta}_i^{(2)})^2\Big]^{1/2} / \sqrt{r(\mathcal{F})}$$

$$\geq \Big[\sum_{i=1}^{M} \sup_{\vec{v} \in V} \frac{|a(\vec{e}, \phi_i \cdot \vec{v}) + b(\phi_i \cdot \vec{v}, \varepsilon)|}{|\phi_i \cdot \vec{v}|_{1,\Omega}}\Big]^{1/2} / \sqrt{r(\mathcal{F})} \geq \hat{\eta}/\sqrt{r(\mathcal{F})}.$$

On the other hand,

$$|\vec{e}|_{1,\Omega} \geq \| \operatorname{div} \cdot \vec{e} \|_{0,\Omega} = \sup_{q \in L_0^2(\Omega)} \frac{|(\operatorname{div} \cdot \vec{e}, q)|}{\|q\|_{0,\Omega}} = \sup_{q \in L_0^2(\Omega)} \frac{|(g - \operatorname{div} \cdot \vec{u}_h, q)|}{\|q\|_{0,\Omega}} = \varsigma.$$

Finally, we obtain

$$|\vec{e}|_{1,\Omega} + \|\varepsilon\|_{0,\Omega} \geq (|\vec{e}|_{1,\Omega} + \|\varepsilon\|_{0,\Omega})/2 + |\vec{e}|_{1,\Omega}/2 \geq (\hat{\eta}/\sqrt{r(\mathcal{F})} + \varsigma)/2.$$

Now we have obtained the error estimator of the mixed finite element method for the Stokes problem. But it is not readily applicable to computation of η and η in Theorems 3.1 and 3.2 in practice. For convenience of computation, we shall present another estimator which is equivalent to that in Theorem 3.1 and 3.2. Here we prove this result only for the case of triangular partition. The proof of the rectangular partition is similar.

Theorem 3.3. *Let $f \in (L^2(\Omega))^2$ and $g \in L^2(\Omega)$ be given functions. Set*

$$\begin{cases} \tau_j^2 = (h_j^2/48) \cdot \iint_{k_j} |\vec{f} + \nabla^2 \vec{u}_h|^2 dx + (h_j/24) \int_{\Gamma_j} |J_{\Gamma_j}[\partial \vec{u}_h/\partial n - p_h \cdot \vec{n}]|^2 ds, \\ \varsigma_j^2 = \iint_{k_j} |g - \operatorname{div} \cdot \vec{u}_h|^2 dx, \quad \theta_j^2 = \tau_j^2 + \beta^{-1} \cdot \varsigma_j^2, \\ \theta^2 = \sum_{j=1}^{m} \theta_j^2 \end{cases} \tag{3.12}$$

where $\Gamma_j = \partial k_j$ is the boundary of $k_j, n = \vec{n}_{\Gamma_j}$ the exterior normal direction and $J_{\Gamma_j}[\,\cdot\,]$ the jump value of some function on Γ_j, i.e.,

$$J_{\Gamma_j}[\partial \vec{u}_h/\partial n - p_h \cdot \vec{n}] = \sum_i [(\partial \vec{u}_h/\partial n|_{k_j} - \partial \vec{u}_h/\partial n|_{k_i^j}) - (p_h|_{k_j} - p_h|_{k_i^j}) \cdot \vec{n}_{\Gamma_j}]$$

where k_i^j denotes the i-th element neighboring the element k_j. Then there exists a constant C_1 independent of $h, [\vec{u}, p], [\vec{u}_h, p_h]$ and Ω, such that

$$|\vec{e}|_{1,\Omega} + |||\varepsilon||| \leq C_1 \cdot \theta. \tag{3.13}$$

If we assume in addition that $\vec{f} \in (P_t(\Delta))^2$ (t is a positive integer), where

$$P_t(\Delta) = \{\varphi \in C(\Omega); \varphi|_{k_j} \in P_t(k_j), j = 1, \cdots, m\}, \tag{3.14}$$

then there exists a constant C_2 depending only on $\sqrt{r(\mathcal{F})}, \sigma, \beta$ and t, such that

$$|\vec{e}|_{1,\Omega} + \|\varepsilon\|_{0,\Omega} \geq C_2 \cdot \theta. \tag{3.15}$$

Proof. First we prove (3.13). It suffices to prove that there exists a constant C such that

$$\eta \leq C \cdot \Big(\sum_{j=1}^{m} \tau_j^2 \Big)^{1/2}.$$

Here we only consider the case that $\eta = \hat{\eta}$. For the other case $\eta = \tilde{\eta}$, the proof is similiar. Let $\Omega_i = \mathrm{Supp}\ {}^0\phi_i$ (Fig. 3.1.). Then we have

$$\mathrm{diam}\ (\Omega_i) \leq 2h_\Delta \leq 2h_j/\gamma,$$
$$\forall\, i = 1, \cdots, M; \quad j = 1, \cdots.$$

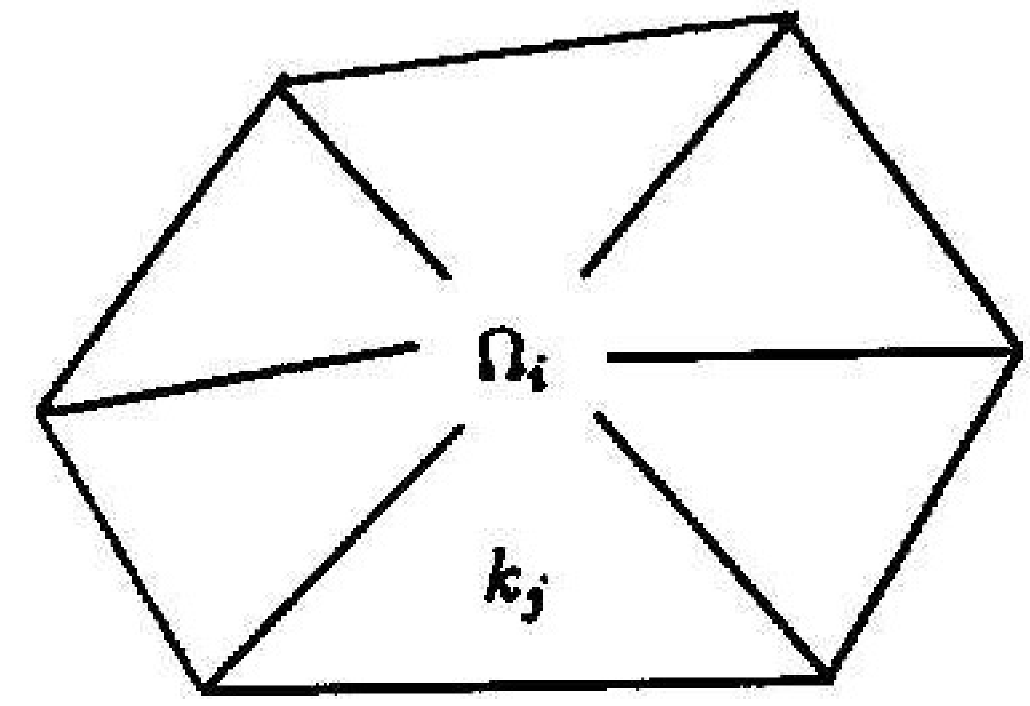

Fig. 3.1

Therefore, for any $\vec{v}$ in $(H_0^1(\Omega_i))^2$, we have

$$\|\vec{v}\|_{0,\Omega_i}^2 \leq 4C'^2\gamma^{-2}h_j^2 \cdot |\vec{v}|_{1,\Omega_i}^2, \quad \sum_{F_j \in \Omega_i} |\vec{v}|_{L(F_j)}^2 \leq C''^2 h_j \cdot |\vec{v}|_{1,\Omega_i}^2 \tag{3.16}$$

there k_j is an element in Ω_i, and F_j the common edge of any two elements contained in Ω_i. The number of F_j in Ω_i is obviously less than or equal to $\chi(\mathcal{F})$.

For any $\vec{v} \in (H^1(\Omega))^2$, Supp ${}^0\vec{v} \subset \Omega_i$,

$$a(\vec{e}, \vec{v}) + b(\vec{v}, \varepsilon) = \iint_{\Omega_i} \vec{f} \cdot \vec{v} dx - \sum_{kj \in \Omega_i} \Big[\iint_{k_j} \nabla \vec{u}_h \cdot \nabla \vec{v} dx - \iint_{k_j} P_h \text{ div } \cdot \vec{v} dx \Big]$$

$$= \iint_{\Omega_i} (\vec{f} + \nabla^2 \vec{u}_h - \vec{\mathrm{grad}} P_h) \cdot \vec{v} dx - \sum_{F_j \in \Omega_i} \int_{F_j} J_{F_j} \Big[\partial \vec{u}_h / \partial n - P_h \cdot \vec{n} \Big] \cdot \vec{v} ds$$

$$\leq \|\vec{f} + \nabla^2 \vec{u}_h\|_{0,\Omega} \cdot \|\vec{v}\|_{0,\Omega_i} + \Big\{ \sum_{F_j \in \Omega_i} \|J_{F_j}[\partial \vec{u}_h/\partial n - P_h \cdot \vec{n}]\|_{L^2(F_j)}^2 \Big\}^{\frac{1}{2}} \Big[\sum_{F_j \in \Omega_i} \|\vec{v}\|_{L^2(F_j)}^2 \Big]^{1/2}$$

$$\leq \Big\{ 2C'' h_j \cdot \|\vec{f} + \nabla^2 \vec{u}\|_{0,\Omega_i}/\gamma + C''\big[\sum_{F_i \in \Omega_i} h_j \|J_{F_j}[\partial \vec{u}_h/\partial n - P_h \cdot \vec{n}]\|_{L^2(F_j)}^2 \big]^{\frac{1}{2}} \Big\} \cdot |\vec{v}|_{1,\Omega_i}.$$

Thus

$$\hat{\eta}^2 = \sum_{i=1}^{m} \hat{\eta}_i^2 \leq 2\Big\{ 4C' \sum_{i=i}^{m} \sum_{k_j \subset \Omega_i} h_j^2 k_j |\vec{f} + \nabla^2 \vec{u}_h|^2 dx/\gamma^2$$

$$+ C'' \sum_{i=1}^{m} \sum_{\substack{k_j \subset \Omega_i \\ \Gamma_j = \partial k_j}} h_j \int_{\Gamma_j} |J_{\Gamma_j}[\partial \vec{u}_h/\partial n - P_h \vec{n}]|^2 ds \Big\}$$

$$\leq 2r(\mathcal{F}) \sum_{i=1}^{m} \Big\{ 4C'^2 \cdot h_j^2 k_j |\vec{f} + \nabla^2 \vec{u}_h|^2 dx + C''^2 \cdot h_j \int_{\Gamma_j} |J_{\Gamma_j}[\partial \vec{u}_h/\partial n - P_h \vec{n}]|^2 ds \Big\}$$

$$\leq C_1^2 \cdot \sum_{i=1}^{m} \tau_j^2 = C_1^2 \cdot \tau^2.$$

Now we prove inequality (3.15). Since

$$\hat{\eta}_i = \sup_{\vec{v}\in(H_0^1(\Omega_i))^2} \frac{\left|(\vec{f}+\nabla^2\vec{u}_h,\vec{v}) + \sum_{F_j\in\Omega_i}\int_{F_j} J_{F_j}[P_h\vec{n}-\partial\vec{u}_h/\partial n]\cdot\vec{v}ds\right|}{|\vec{v}|_{1,\Omega_i}}.$$

For any given $F_j\subset\Omega_i$, let $F_j=k_1\cap k_2$, where k_1 and k_2 are contained in Ω_i, and let l_1 and l_2 be the other two sides of k_2 (Fig. 3.2).

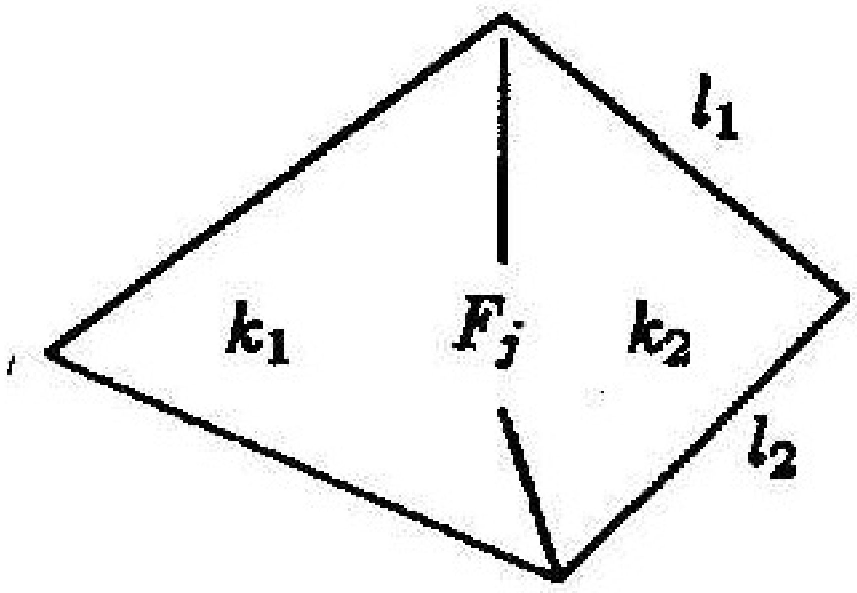

Fig. 3.2

Let

$$\vec{R}=\vec{f}+\nabla^2\vec{u}_h,$$

and

$$V_{\vec{R}}=\{\vec{v}\in(H_0^1(k_1\cup k_2))^2, \iint_{k_1\cup k_2}\vec{R}\cdot\vec{v}dx=0\}.$$

It is easy to prove that $V_{\vec{R}}$ is non-empty and $V_{\vec{R}}\subset(H_0^1(\Omega))^2$. Thus

$$\hat{\eta}_i \ge \sup_{\vec{v}\in V_{\vec{R}}} \frac{\int_{F_j} J_{F_j}[P_h\vec{n}-\partial\vec{u}_h/\partial n]\cdot\vec{v}ds|}{|\vec{v}|_{1,k_1\cup k_2}} \ge \sup_{\substack{\vec{v}\in V_{\vec{R}}\\ |\vec{v}|_{1,k_1}\ge|\vec{v}|_{1,k_2}}} \frac{\left|\int_{F_j} J_{F_j}[P_h\vec{n}-\partial\vec{u}_h/\partial n]\cdot\vec{v}ds\right|}{|\vec{v}|_{1,k_1,\cup k_2}}$$

$$\ge \frac{1}{\sqrt{2}}\iint_{k_2} \sup_{\substack{\vec{R}\cdot\vec{v}dx=0\\ \vec{v}|_{l_1\cup l_2}=0}} \frac{\left|\int_{F_j} J_{F_j}[P_h\vec{n}-\partial\vec{u}_h/\partial n]\cdot\vec{v}ds\right|}{|\vec{v}|_{1,k_2}}.$$

Let $\vec{S}=J_{F_j}[P_h\vec{n}-\partial\vec{u}_h/\partial n]$. By an affine transformation one can transform k_2 to $\hat{k}$ which is the standard element. The transformed functions and variables are denoted by the superscript "∧".

Let

$$\xi(\hat{\vec{S}},\hat{\vec{R}}) = \sup_{\substack{\iint\hat{\vec{R}}\cdot\hat{\vec{v}}d\hat{x}=0\\ \hat{v}|_{\hat{l}_1\cup\hat{l}_2}=0}} \frac{\left|\int_{\hat{F}}\hat{\vec{v}}\cdot\hat{\vec{S}}d\hat{s}\right|}{|\hat{\vec{v}}|_{1,\hat{k}}\|\hat{\vec{s}}\|_{L^2(\hat{F})}} > 0.$$

Since $\hat{\vec{S}}$ and $\hat{\vec{R}}$ are in the finite dimensional space $(P_t(\hat{k}))^2$, the minimum value ξ_0 of $\xi(\hat{\vec{S}},\hat{\vec{R}})$ in $(P_t(\hat{k}))^2$ can be reached. Therefore

$$\xi(\hat{\vec{S}},\hat{\vec{R}})\ge\xi_0>0$$

where $\xi_0>0$ depends only on t.

By an affine transformation we can turn $\hat{k}$ back to k_2. Thus we have

$$\hat{\eta}_i \ge C\cdot h_j^{1/2}\cdot\|J_{F_j}[R_h\cdot\vec{n}-\partial\vec{u}_h/\partial n]\|_{L^2(F_j)}.$$

Since the number of $F_j \subset \Omega_i$ is not greater than $\tau(\mathcal{T})$, we have

$$\hat{\eta}_i^2 \geq C^2\Big\{ \sum_{F_j \subset \Omega_i} h_j \|J_{F_j}[P_h\vec{n} - \partial\vec{u}_h/\partial n]\|^2_{L^2(F_j)}\Big\}/\tau(\mathcal{T}).$$

On the other hand, since $\vec{f} + \nabla^2\vec{u}_h \in (P_t(\Delta))^2$, one can prove that

$$\hat{\eta}_i^2 \geq C^2\Big\{ \sum_{F_j \subset \Omega_i} h_j^2 \cdot \iint_{k_j} |\vec{f} + \nabla^2\vec{u}_h|^2 dx\Big\}/\tau(\mathcal{T})$$

(see [10]). Therefore

$$\hat{\eta}^2 = \sum_{i=1}^m \hat{\eta}_i^2 \geq C^2\Big\{ \sum_{i=1}^m \sum_{k_j \subset \Omega_i} (h_j^2/48)\|\vec{f} + \nabla^2\vec{u}_h\|^2_{0,k_j}$$

$$+\sum_{i=1}^m \sum_{F_j \subset \Omega_i} (h_j/24)\|J_{F_j}[P_h\vec{n} - \partial\vec{u}_h/\partial n]\|^2_{L^2(F_j)}\Big\} \geq C \cdot \sum_{i=1}^m \tau_j^2.$$

According to Therorem 3.2, finally we can obtain (3.15).

Remark 1. It seems that the assumption $\vec{f} \in (P_t(\Delta))^2$ is not natural in Theorem 3.3. But in computing a posteriori error estimator and getting a finite element solution, the computation of the integral value of $\vec{f}(x)$ is involuved. In practice, however, for general function $\vec{f}(x)$ we always compute the integral value by the Gauss quadrature formulas. In fact, at this time one has assumed that $\vec{f}(x) \in (P_t(\Delta))^2$.

Remark 2. In Theorem 3.3, the numbers 48 and 24 in (5.12) are determined by means of the results of the estimator, obtained by Babuska, of the finite element method for the Dirichlet problem of Poisson equation. The results of numerical examples show also that if we take such coefficients in (3.12), the constants C_1 and C_2 in (3.13) and (3.15) will be near to 1. Such a result is just what we wish for.

§4. Numerical Examples

In this section, we present two numerical examples which support the above theorecal results. We employ the conforming square bi-quadratic elements for $\vec{u}_h$ and constant elements for p_h in both examples.

Example 1. In the problem, $\nu = 1$, $\Omega = \{(x,y);\ 0 \leq x, y \leq 1\}$ (see Fig. 4.1), and

$$\vec{f}(x,y) = (f_1(x,y), f_2(x,y))^T,$$

$$g(x,y) = -\frac{1}{3} - x^{-1/3}y^{-1/3}[y^2(5x-2)(y-1) + x^2(5y-2)(x-1)],$$

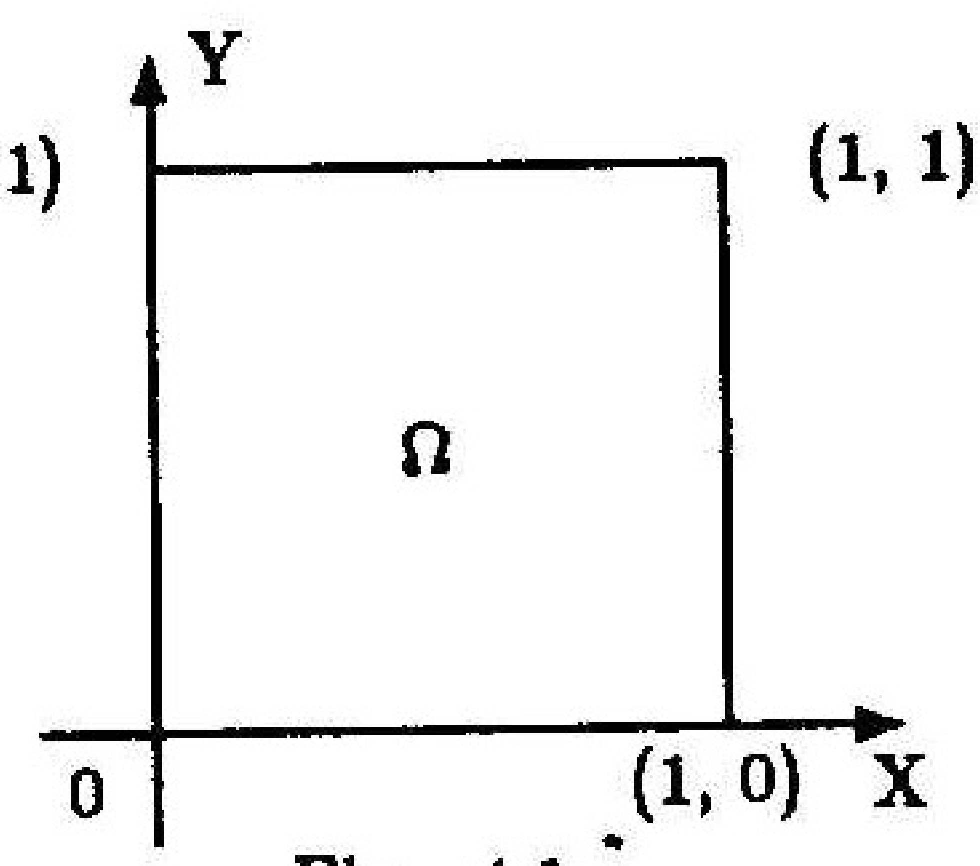

Fig. 4.1

where

$$f_1(x,y) = -\frac{2}{9} - x^{-1/3}y^{-1/3}[5y^2(1-y) - x^2(20y+3) + 20xy],$$

$$f_2(x,y) = -\frac{2}{9} - x^{-1/3}y^{-1/3}[5x^2(1-x) - y^2(20x+3) + 20xy].$$

For this problem there exists an analytic solution:

$$\vec{u}(x,y) = (u_1(x,y), u_2(x,y))^T,$$
$$u_1(x,y) = x^{2/3}(x-1)y^{5/3}(y-1), \quad u_2(x,y) = x^{5/3}(x-1)y^{2/3}(y-1),$$
$$p(x,y) = -(2/3)[x^{-1/3}y^{5/3}(y-1) + x^{5/3}y^{-1/3}(x-1)] - 9/44.$$

Example 1 is a Stokes problem of which the pressure solution and the partial derivative of the velocity solution are singular at (0, 0). We can compare the true error of the finite element solution with a posteriori error estimator presented in Theorem 3.3. Let $\rho = (|\vec{e}|_{1,\Omega} + \|\epsilon\|_{0,\Omega})/\eta$. Then we have the following table.

m	$\lvert\vec{e}\rvert_{1,\Omega} + \|\epsilon\|_{0,\Omega}$	η	ρ
16	$0.64326386 \times 10^{-1}$	$0.70648597 \times 10^{-1}$	1.0983
25	$0.59469328 \times 10^{-1}$	$0.69790280 \times 10^{-1}$	1.1736
36	$0.55920162 \times 10^{-1}$	$0.68078825 \times 10^{-1}$	1.2174

In the table we can see that the constants C_1 and C_2 in Theorem 3.3 are not large since ρ is not large.

Example 2. Solve problem (2.1), where

$$\Omega\{(x,y); 0 \le x \le 0.5 \text{ and } 0 \le y \le 1, \text{ or } 0.5 \le x \le 1 \text{ and } 0.5 \le y \le 1\},$$

(see Fig. 4.2) and

$$f_1(x,y) = f_2(x,y) = 1, \quad g(x,y) = 0.$$

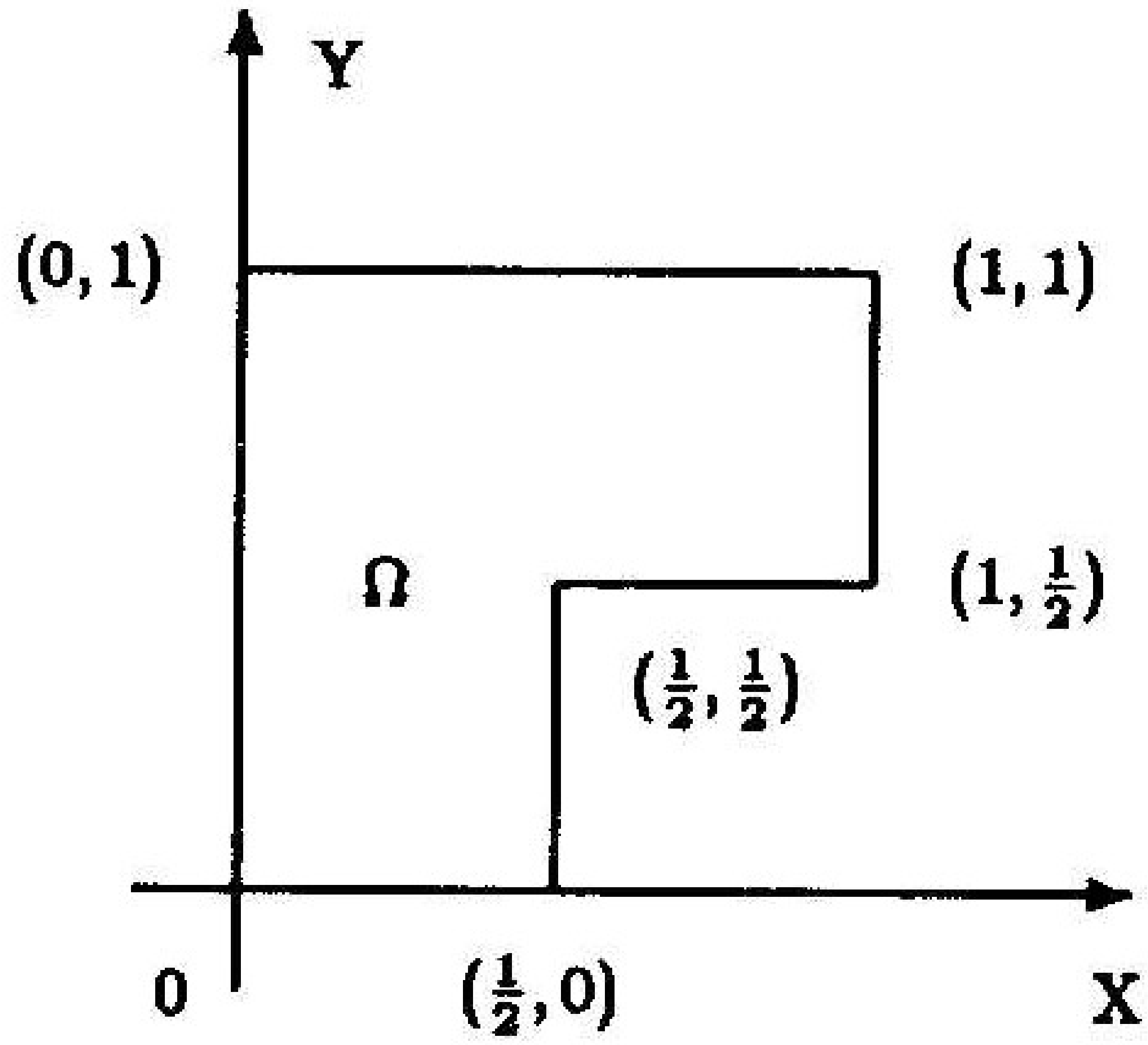

Fig. 4.2

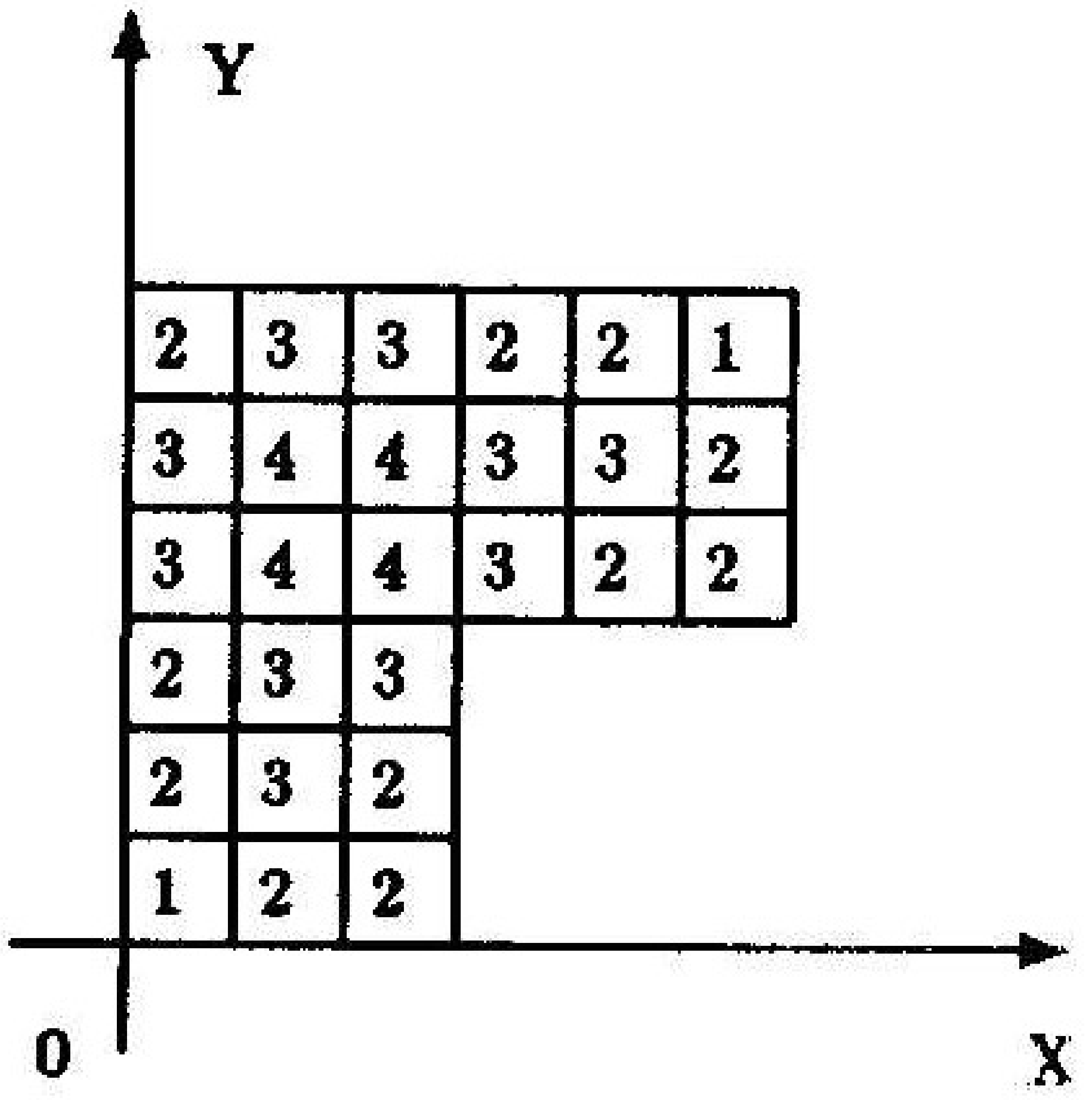

Fig. 4.3

Example 2 is a Stokes problem for which there exists a concave angle in its domain Ω. The analytic solution does not exist for the problem. But, in general, the solution $[\vec{u}, p]$ changes rapidly at the neighborhood of the point (0.5, 0.5) where the concave angle appears. Hence, the errors in the elements which neighbor the point (0.5, 0.5) are larger. We separate the error indicators at all elements computed by means of theorem 3.3 into four classes. The error indicators at the elements in Class L are between $0.09 \times L$ to $0.09 \times (L+1)$. Figure 4.3 illustrates the distribution of the elements of the various classes. From Figure 4.3 we can see that the elements at which the error indicators are larger are generally in the neighborhood of the concave angle point (0.5, 0.5). This shows that the error indicator approximately equals the true error at each element.

Both examples show that the posteriori error estimator presented in Section 3 is a better estimator.

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