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BLOCK IMPLICIT HYBRID ONE-STEP METHODS*

Miao Jian-ming
(Shanghai Normal University, Shanghai, China)

Abstract

A class of k-block implicit hybrid methods for solving the initial value problem for ordinary differential equations are studied, which take a block of k new values at each step. These methods are examined for the property of A-stability. It is shown that the method of order $2k+2$ exists uniquely, and these methods are A-stable for block sizes $k=1,2,\ldots,5$.

§1. Introduction

We shall study a class of methods for solving numerically the initial value problem for ordinary differential equations. These methods are named k-block implicit hybrid one-step methods, and take k new values at each step.

Block methods have been studied by a number of authors, such as Rosser, Shampine and Watts, Bichart and Picel, and Zhou Bing. Shampine and Watts[6], [7] did further research on theori s of block methods. They presented a different approach based on interpolatory formulas of Newton-Cotes type; the methods are of order $k+1$ for k odd and $k+2$ for k even. They also showed that the methods are A-stable for sizes $k=1,2,\ldots,8$.

The fatal defect of block methods is inversion of a $km \times km$ matrix during Newton iterations, where m is the number of differential equations. So the use of higher order block methods is limited. To avoid the defect, we present a class of block implicit hybrid one-step methods, which are combinations of hybrid methods with block methods. These methods with small k possess higher accuracy and good stability. It is shown that the method of order $2k+2$ exists uniquely, and these methods are A-stable for block sizes $k=1,2,\ldots,5$.

§2. A General Formulation and Convergence

Consider the initial value problem

$$y' = f(x,y), \quad y(\alpha) = \eta, \quad \alpha \le x \le \beta. \tag{2.1}$$

Let $x_{n+i} = x_n + ih, x_{n+v_i} = x_n + v_i h$, where $n = mk, m = 0,1,2,\ldots, i = 1,2,\ldots,k$, and $v_i \notin Z, i = 1,2,\ldots,k, v_1 < v_2 < \ldots < v_k$. Let y_j be the approximation of $y(x_j)$. Then the formulas are in the form

$$\begin{cases} Y_m = y_n K^0 + hBF(Y_m) + hf_n b + hDF(Y_{m+v}), \\ Y_{m+v} = -A_* Y_m - y_n a_* + hB_* F(Y_m) + hf_n b_*, \end{cases} \tag{2.2}$$

where $f_j = f(x_j, y_j), k^0 = (1,\ldots,1)^T, B, D, A_*, B_* \in R^{k\times k}, b, a_*, b_* \in R^{k\times 1}$, D is nonsingular, $Y_m = (y_{n+1},\ldots,y_{n+k})^T, Y_{m+v} = (y_{n+v_1},\ldots,y_{n+v_k})^T, F(Y_m) = (f_{n+1},\ldots,f_{n+k})^T$, $F(Y_{m+v}) = (f_{n+v_1},\ldots,f_{n+v_k})^T$.

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Equation (2.2) is a system of nonlinear equations for Y_m and can be written as

$$Y_m = y_n K^0 + hBF(Y_m) + hf_n b + hDF(-A_* Y_m - y_n a_* + hB_* F(Y_m) + hf_n b_*) \equiv G(Y_m);$$

thus

$$G'(Y_m) = h[BF'(Y_m) + DF'(Y_{m+v})(-A_* + hB_* F'(Y_m)].$$

If h is suitably small, we have $\|G'(Y_m)\| < 1$. Then (2.2) has a unique solution. In practice, we may have to presume the existence of a solution.

With the method (2.2), we define two linear difference operator vectors $\mathcal{L}$ and $\mathcal{L}^*$ by

$$\mathcal{L}[Y_m(x); h] = Y_m(x) - y(x)K^0 - hBY_m'(x) - hy'(x)b - hDY_{m+v}'(x), \tag{2.3}$$

$$\mathcal{L}^*[Y_m(x); h] = Y_{m+v}(x) + A_* Y_m(x) + y(x)a_* - hB_* Y_m'(x) - hy'(x)b_*, \tag{2.4}$$

where $Y_m^{(i)}(x) = (y^{(i)}(x+h), \cdots, y^{(i)}(x+kh))^T, Y_{m+v}^{(i)}(x) = (y^{(i)}(x+v_1 h), \cdots, y^{(i)}(x+v_k h))^T, i = 0, 1$. Expanding $y(x+ih), y(x+v_i h)$ and their derivatives as Taylor series about x and collecting terms in (2.3) and (2.4) give

$$\mathcal{L}[Y_m(x); h] = y(x)c_0 + hy'(x)c_1 + \cdots + h^p y^{(p)}(x)c_p + \ldots, \tag{2.5}$$

$$\mathcal{L}^*[Y_m(x); h] = y(x)c_0^* + hy'(x)c_1^* + \ldots + h^q y^{(q)}(x)c_q^* + \ldots, \tag{2.6}$$

where c_p and c_q^* are constant vectors. Comparing (2.3) and (2.4) with (2.5) and (2.6), we have

$$\begin{cases} c_0 = 0, \\ c_1 = K - BK^0 - b - Dv^0, \\ c_p = K^p/p! - BK^{p-1}/(p-1)! - Dv^{p-1}/(p-1)!, \quad p = 2, 3, \ldots, \end{cases} \tag{2.7}$$

$$\begin{cases} c_0^* = v^0 + A_* K^0 + a_*, \\ c_1^* = v + A_* K - B_* K^{0} - b_*, \\ c_q^* = v^q/q! + A_* K^q/q! - B_* K^{q-1}/(q-1)!, \quad q = 2, 3, \ldots, \end{cases} \tag{2.8}$$

where $K^s = (1^s, 2^s, \ldots, k^s)^T$ and $v^s = (v_1^s, v_2^s, \ldots, v_k^s)^T$. For formula (2.2), a convergence theorem can be easily obtained.

Theorem 1. *Suppose the method is defined by (2.2), and the linear difference operator vectors $\mathcal{L}$ and $\mathcal{L}^*$ satisfy $\|\mathcal{L}\| = O(h^{p+1})$ and $\|\mathcal{L}^*\| = O(h^{q+1})$. Then the method is convergent with global error of order h^r where $r = min(p, q+1)$, and the method is said to be of order r.*

In order to obtain a high order method, we choose $B, D, A_*, B_*, b, v, a_*, b_*$, as follows:

$$b = K - BK^0 - Dv^0, \tag{2.9a}$$

$$K^p/p! - BK^{p-1}/(p-1)! - Dv^{p-1}/(p-1)! = 0, \quad p = 2, 3, \ldots, 2k+2; \tag{2.9b}$$

$$\begin{cases} a_* = -v^0 - A_* K^0, \\ b_* = v + A_* K - B_* K^0, \end{cases} \tag{2.10a}$$

$$v^q/q! + A_* K^q/q! - B_* K^{q-1}/(q-1)! = 0, \quad q = 2, 3, \ldots, 2k+1. \tag{2.10b}$$

Then we have

Theorem 2. *The method (2.2) of order $2k+2$ exists uniquely.*

Proof. It is sufficient to prove that solutions of (2.9b) and (2.10b) exist uniquely. Since equations (2.9b) and (2.10b) are nonlinear, there are some troubles. However, if we can determine v such that $v_i \neq j, i = 1, 2, \ldots, k, j = 0, 1, \ldots, k$, and $v_i < v_j$ when $i < j$ (the determination of v will be given in §3), then substituting v into the first $2k$ equations of (2.9b) we obtain a system of equations whose coefficient matrix is a Vandermonde matrix. Hence B, D are determined uniquely. Substituting v into (2.10b) gives

$$(A, -B_*)\begin{pmatrix} K^2 & K^3 & \cdots\cdots & K^{2k+1} \\ 2K & 3K^2 & \cdots & (2k+1)K^{2k} \end{pmatrix} = -(v^2, v^3, \ldots, v^{2k+1}). \tag{2.11}$$

Let

$$X = \begin{pmatrix} K^2 & K^3 & \cdots\cdots & K^{2k+1} \\ 2K & 3K^2 & \cdots & (2k+1)K^{2k} \end{pmatrix}. \quad and \quad z = (z_1, \ldots, z_{2k})^T.$$

If $Xz = 0$, then

$$\begin{cases} z_1K^2 + z_2K^3 + \ldots + z_{2k}K^{2k+1} = 0. \\ 2z_1K + 3z_2K^2 + \ldots + (2k+1)z_{2k}K^{2k} = 0. \end{cases} \tag{2.12}$$

Let $h(x)$ be a polynomial

$$h(x) = z_1x^2 + z_2x^3 + \ldots + z_{2k}x^{2k+1}. \tag{2.13}$$

Then, from (2.12) we have $h(j) = h'(j) = 0, j = 0, 1, \ldots, k$. Thus the polynomial $h(x)$ has at least $2k+2$ zeros, and so $z_1 = z_2 = \ldots = z_{2k} = 0$. Hence X is nonsingular, and A_*, B_* are determined uniquely.

§3. Numerical Stability

When formula (2.2) is applied to the test equation $y' = \lambda y$, Re $\lambda < 0$, it is of the form

$$(I - \bar{h}B + \bar{h}DA_* - \bar{h}^2DB_*)Y_m = y_n(K^0 + \bar{h}b - \bar{h}Da_* + \bar{h}^2Db_*) \tag{3.1}$$

where $\bar{h} = \lambda h$. Let

$$x(\bar{h}) = (I - \bar{h}B + \bar{h}DA_* - \bar{h}^2DB_*)^{-1}(K^0 + \bar{h}b - \bar{h}Da_* + \bar{h}^2Db_*), \tag{3.2}$$

where $x(\bar{h}) = (\xi_1(\bar{h}), \ldots, \xi_k(\bar{h}))^T$. Then we have

$$\begin{cases} y_{n+k} = \xi_k(\bar{h})y_n = [\xi_k(\bar{h})]^{m+1}y_0, \\ y_{n+j} = \xi_j(\bar{h})y_n = \xi_j(\bar{h})[\xi_k(\bar{h})]^m y_0, \ j \neq k. \end{cases} \tag{3.3}$$

Definition. *The block implicit hybrid method (2.2) is said to be absolutely stable for $\bar{h}$ if $|\xi_k(\bar{h})| < 1$. The region of absolute stability is defined as the set $S = \{\bar{h} \mid |\xi_k(\bar{h})| < 1\}$. The method (2.2) is said to be A-stable if $C^- \subset S$.*

In order to obtain the explicit expression of $x(\bar{h})$, using Cramer's rule, we can rewrite x as

$$x(\bar{h}) = \sum_{i=0}^{2k} p_i\bar{h}^i \Big/ \sum_{i=0}^{2k} r_i\bar{h}^i, \ r_0 = 1, \tag{3.4}$$

where $p_i = (p_i^{(1)}, \dots, p_i^{(k)})^T$. Multiplying by $\sum_{i=0}^{2k} r_i \bar{h}^i (I - \bar{h}B + \bar{h}DA_* - \bar{h}^2 DB_*)$ on both sides of (3.2) from left, and comparing coefficients in $\bar{h}^i$, we obtain

$$p_0 = r_0 K^0, \tag{3.5a}$$

$$p_1 + (DA_* - B)p_0 = r_1 K^0 + r_0(b - Da_*), \tag{3.5b}$$

$$p_{i+1} + (DA_* - B)p_i - DB_* p_{i-1} = r_{i+1}K^0 + r_i(b - Da_*) + r_{i-1}Db_*, i = 1, 2, \dots, 2k-1, \tag{3.5c}$$

$$(DA_* - B)p_{2k} - DB_* p_{2k-1} = r_{2k}(b - Da_*) + r_{2k-1}Db_*, \tag{3.5d}$$

$$DB_* p_{2k} = -r_{2k} Db_*. \tag{3.5e}$$

Eliminating v from (2.9) and (2.10), we have

$$K + (DA_* - B)K^0 + Da_* - b = 0, \tag{3.6a}$$

$$K^2/2! + (DA_* - B)K - DB_* K^0 - Db_* = 0, \tag{3.6b}$$

$$K^{p+2}/(p+2)! + (DA_* - B)K^{p+1}/(p+1)! - DB_* K^p/p! = 0, \quad p = 1, 2, \dots, 2k. \tag{3.6c}$$

Then we can determine p_i, r_i from (3.5) and (3.6).

Lemma 1. *If the method* (2.2) *is defined by* (2.9) *and* (2.10), *then*

$$\text{(i)} \quad p_i = \sum_{s=0}^{i} r_{i-s} K^s/s!, \quad i = 0, 1, \dots, 2k, \tag{3.7}$$

$$\text{(ii)} \quad r_i = (2k - i + 1)(2k - i + 2)\varphi^{(2k-i)}(0)/(2k+2)!, \quad i = 0, 1, \dots, 2k, \tag{3.8}$$

where

$$\varphi(x) = [(x-1)(x-2)\dots(x-k)]^2. \tag{3.9}$$

Proof. Since $r_0 = 1$, then $p_0 = K^0$. From (3.5b) and (3.6a), we have

$$p_1 = -(DA_* - B)K^0 + b - Da_* + r_1 K^0 = K + r_1 K^0.$$

Suppose (3.7) is true for $i \le 2k - 1$. Then for $i + 1$ we have

$$\begin{aligned} p_{i+1} &= -(DA_* - B)\sum_{s=0}^{i} r_{i-s}K^s/s! + DB_* \sum_{s=0}^{i-1} r_{i-s-1}K^s/s! + r_{i-1}Db_* + r_i(b - Da_*) + r_{i+1}K^0 \\ &= r_i[-(DA_* - B)K^0 + b - Da_*] + r_{i-1}[-(DA_* - B)K + DB_* K^0 + Db_*] \\ &\quad + \sum_{s=1}^{i-1} r_{i-s-1}[-(DA_* - B)K^{s+1}/(s+1)! + DB_* K^s/s!] + r_{i+1}K^0 \\ &= r_i K + r_{i-1}K^2/2! + \sum_{s=1}^{i-1} r_{i-s-1}K^{s+2}/(s+2)! + r_{i+1}K^0 = \sum_{s=0}^{i+1} r_{i+1-s}K^s/s! \end{aligned}$$

Thus (3.7) holds for $i \le 2k$. From (3.5d) we have

$$(DA_* - B)\sum_{s=0}^{2k} r_{2k-s}K^s/s! - DB_* \sum_{s=0}^{2k-1} r_{2k-1-s}K^s/s! = r_{2k}(b - Da_*) + r_{2k-1}Db_*.$$

That is

$$r_{2k}[-(DA_* - B)K^0 + b - Da_*] + r_{2k-1}[-(DA_* - B)K + DB_*K^0 + Db_*]$$
$$+ \sum_{s=1}^{2k-1} r_{2k-1-s}[-(DA_* - B)K^{s+1}/(s+1)! + DB_*K^s/s!]$$
$$= r_{2k}K + r_{2k-1}K^2/2! + \sum_{s=1}^{2k-1} r_{2k-1-s}K^{s+2}/(s+2)! = 0.$$

Then we have

$$\sum_{s=0}^{2k} r_{2k-s}K^{s+1}/(s+1)! = 0. \tag{3.10}$$

From (3.5e) we can have

$$\sum_{s=0}^{2k} r_{2k-s}K^{s+2}/(s+2)! + (DA_* - B)\sum_{s=0}^{2k} r_{2k-s}K^{s+1}/(s+1)! = 0.$$

Then from (3.10) we have

$$\sum_{s=0}^{2k} r_{2k-s}K^{s+2}/(s+2)! = 0. \tag{3.11}$$

Let

$$g(x) = \sum_{s=0}^{2k} r_{2k-s}x^{s+2}/(s+2)! \tag{3.12}$$

From (3.10) and (3.11) we have $g(j) = g'(j) = 0, j = 0, 1, \ldots, k$; hence

$$g(x)/x^2 = \sum_{s=0}^{2k} r_{2k-s}x^s/(s+2)! = \varphi(x)/(2k+2)!$$

and so

$$r_{2k-s} = (s+2)!\varphi^{(s)}(0)/(2k+2)!s! = (s+1)(s+2)\varphi^{(s)}(0)/(2k+2)!$$

Let $i = 2k - s$. Then (3.8) holds.

In fact, we can also determine v uniquely. From (3.5e) and (2.10) we have

$$r_{2k}D(v + A_*K) + DB_*\sum_{s=1}^{2k} r_{2k-s}K^s/s! = r_{2k}D(v + A_*K)$$
$$+ D\sum_{s=1}^{2k} r_{2k-s}[v^{s+1}/(s+1)! + A_*K^{s+1}/(s+1)!]$$
$$= D\sum_{s=0}^{2k} r_{2k-s}v^{s+1}/(s+1)! + DA_*\sum_{s=0}^{2k} r_{2k-s}K^{s+1}/(s+1)! = 0.$$

By using (3.10), we have

$$\sum_{s=0}^{2k} r_{2k-s}v^{s+1}/(s+1)! = 0. \tag{3.13}$$

Hence $v_j (j = 1, 2, \ldots, k)$ are k zeros of $[\varphi(x)x^2]' = 2x(x-1)\cdots(x-k)[x(x-1)\cdots(x-k)]'$; then $v_j (j = 1, 2, \ldots, k)$ are k zeros of $[x(x-1)\cdots(x-k)]'$. It can be easily seen that $v_i \in (i-1, i), i = 1, 2, \ldots, k$, so Theorem 2 holds.

In order to consider the numerical stability, we write $\xi_k(\bar{h})$ as

$$\xi_k(\bar{h}) = \sum_{i=0}^{2k} p_i^{(k)} \bar{h}^i \Big/ \sum_{i=0}^{2k} r_i \bar{h}^i \equiv P(\bar{h})/R(\bar{h}). \tag{3.14}$$

Then we have Lemma 2.

Lemma 2.

$$p_i^{(k)} = (-1)^i r_i, \quad i = 0, 1, \ldots, 2k. \tag{3.15}$$

Proof. From (3.7) we have

$$p_i^{(k)} = \sum_{s=0}^{i} r_{i-s} k^s / s!$$

by (3.8),

$$\begin{aligned} p_i^{(k)} &= \sum_{s=0}^{i} (2k - i + s + 1)(2k - i + s + 2)\varphi^{(2k-i+s)}(0) k^s / (2k+2)! s! \\ &= \sum_{s=0}^{i} [(2k-i+1)(2k-i+2) + 2(2k-i+2)s + s(s-1)] \frac{\varphi^{(2k-i+s)}(0) k^s}{(2k+2)! s!} \\ &= [(2k-i+1)(2k-i+2)\varphi^{(2k-i)}(k) + 2(2k-i+2)k\varphi^{(2k-i+1)}(k) \\ &\quad + k^2 \varphi^{(2k-i+2)}(k)]/(2k+2)! \end{aligned}$$

Take $2k - i + 2$ derivatives on both sides of the equality $(x-k)^2 \varphi(k-x) = \varphi(x)x^2$ and put $x = 0$. Then we have

$$\begin{aligned} &(2k-i+1)(2k-i+2)\varphi^{(2k-i)}(k) + 2(2k-i+2)k\varphi^{(2k-i+1)}(k) + k^2\varphi^{(2k-i+2)}(k) \\ &\quad = (-1)^i (2k-i+1)(2k-i+2)\varphi^{(2k-i)}(0). \end{aligned} \tag{3.16}$$

By use of (3.16), $p_i^{(k)}$ becomes

$$p_i^{(k)} = (-1)^i (2k-i+1)(2k-i+2)\varphi^{(2k-i)}(0)/(2k+2)! = (-1)^i r_i, i = 0, 1, \ldots, 2k.$$

Hence $R(\bar{h}) \equiv P(-\bar{h})$. Then we have

Theorem 3. *If the zeros of the polynomial $P(\bar{h})$ are all in the left-plane C^-, then the block implicit hybrid method (2.2) is A-stable.*

Proof. Since the zeros of $P(\bar{h})$ are all in $C^-, P(-\bar{h})$ has no zero in C^-; hence $\xi_k(\bar{h})$ is analytic in C^-. From $|\xi_k(iy)| = 1, y \in (-\infty, \infty), i = \sqrt{-1}$, and $|\xi_k(\bar{h})| \to 1$ as $|\bar{h}| \to \infty$, by using the maximum modulus principle, we have $|\xi_k(\bar{h})| < 1, \bar{h} \in C^-$. This completes the proof.

Lemma 3. *Let $z_k^{(l)}, 1 \le l \le 2k$, be the zeros of the polynomial $P(z)$. Then*

$$\mathrm{Re} z_k^{(l)} < 0, \quad 1 \le l \le 2k, k = 1, 2, \ldots, 5. \tag{3.17}$$

Proof. Decompose $P(z)$ into two polynomials $E(z)$ and $F(z)$, which contain respectively only the even and odd terms of $P(z)$. Then, with $g(z) \equiv E(z)/F(z)$, it follows that

$$P(z)/F(z) = g(z) + 1.$$

We expand the function $g(z)$ into fractions:

$$g(z) = a_0 z + \cfrac{1}{a_1 z + \cfrac{1}{a_2 z \, \ddots + \cfrac{1}{a_t z}}}$$

By calculation, we have $a_i > 0$ for $k = 1, 2, \ldots, 5$. Since the coefficients of the fraction are positive real numbers, if $\mathrm{Re}z \geq 0$, we have $\mathrm{Re}g(z) \geq 0$. Thus $\mathrm{Re}[P(z)/F(z)] \geq 1$; hence $P(z) \neq 0$.

Theorem 4. *Block implicit hybrid methods* (2.2) *defined by* (2.9) *and* (2.10) *are A-stable for block sizes* $k = 1, 2, \ldots, 5$.

The coefficient matrices and vectors are displayed in Table 1 for $k \leq 3$.

Table 1

k	b	a_*	b_*	v	B
1	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{6}$
2	$\begin{matrix}\frac{31}{240}\\ \frac{2}{15}\end{matrix}$	$\begin{matrix}\frac{-5-2\sqrt{3}}{18}\\ \frac{-5+2\sqrt{3}}{18}\end{matrix}$	$\begin{matrix}\frac{3+\sqrt{3}}{54}\\ \frac{3-\sqrt{3}}{54}\end{matrix}$	$\begin{matrix}\frac{3-\sqrt{3}}{3}\\ \frac{3+\sqrt{3}}{3}\end{matrix}$	$\begin{matrix}\frac{4}{15} & \frac{1}{240}\\ \frac{8}{15} & \frac{2}{15}\end{matrix}$
3	$\begin{matrix}\frac{106}{945}\\ \frac{107}{945}\\ \frac{4}{35}\end{matrix}$	$\begin{matrix}-\frac{27+10\sqrt{5}}{108}\\ -\frac{13}{512}\\ \frac{-27+10\sqrt{5}}{108}\end{matrix}$	$\begin{matrix}\frac{1}{24}+\frac{\sqrt{5}}{72}\\ \frac{3}{512}\\ \frac{1}{24}-\frac{\sqrt{5}}{72}\end{matrix}$	$\begin{matrix}\frac{3-\sqrt{5}}{2}\\ \frac{3}{2}\\ \frac{3+\sqrt{5}}{2}\end{matrix}$	$\begin{matrix}\frac{151}{420} & \frac{11}{420} & \frac{1}{945}\\ \frac{58}{105} & \frac{23}{105} & \frac{2}{945}\\ \frac{81}{140} & \frac{81}{140} & \frac{4}{35}\end{matrix}$

k	D	A_*	B_*
1	$\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{8}$
2	$\begin{matrix}\frac{3}{10}+\frac{3\sqrt{3}}{16} & \frac{3}{10}-\frac{3\sqrt{3}}{16}\\ \frac{3}{5} & \frac{3}{5}\end{matrix}$	$\begin{matrix}-\frac{4}{9} & -\frac{5-2\sqrt{3}}{18}\\ -\frac{4}{9} & -\frac{5+2\sqrt{3}}{18}\end{matrix}$	$\begin{matrix}-\frac{4\sqrt{3}}{27} & \frac{-3+\sqrt{3}}{54}\\ \frac{4\sqrt{3}}{27} & \frac{-3-\sqrt{3}}{54}\end{matrix}$
3	$\begin{matrix}\frac{41}{140}+\frac{2\sqrt{5}}{15} & -\frac{16}{189} & \frac{41}{140}-\frac{2\sqrt{5}}{15}\\ \frac{2}{7}+\frac{2\sqrt{5}}{15} & \frac{512}{945} & \frac{2}{7}-\frac{2\sqrt{5}}{15}\\ \frac{81}{140} & \frac{16}{35} & \frac{81}{140}\end{matrix}$	$\begin{matrix}-\frac{1}{4} & -\frac{1}{4} & \frac{-27+10\sqrt{5}}{108}\\ -\frac{243}{512} & -\frac{243}{512} & -\frac{13}{512}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{-27-10\sqrt{5}}{108}\end{matrix}$	$\begin{matrix}\frac{-1-\sqrt{5}}{8} & \frac{1-\sqrt{5}}{8} & \frac{-3+\sqrt{5}}{72}\\ \frac{81}{512} & -\frac{81}{512} & -\frac{3}{512}\\ \frac{-1+\sqrt{5}}{8} & \frac{1+\sqrt{5}}{8} & \frac{-3-\sqrt{5}}{72}\end{matrix}$

§4. Numerical Examples

When we apply the Newton iteration to solve the nonlinear equations (2.2), the following matrix needs inversing during the iteration:

$$Q = I - hB\frac{\partial F(Y_m)}{\partial Y_m} - hD\frac{\partial F(Y_{m+v})}{\partial Y_m} - \left(A_* + hB_*\frac{\partial F(Y_m)}{\partial Y_m}\right), \tag{4.1}$$

where

$$\frac{\partial F(Y_m)}{\partial Y_m} = \mathrm{diag}(J_{n+1}, \ldots, J_{n+k}), \quad \frac{\partial F(Y_{m+v})}{\partial Y_m} = \mathrm{diag}(J_{n+v_1}, \ldots, J_{n+v_k}), \tag{4.2}$$

and

$$J_i = \frac{\partial f}{\partial y}(x_i, y_i). \tag{4.3}$$

When (2.1) is a system containing m differential equations, then Q is a matrix of order km; if k is large, much work should be done on inversing Q. Therefore, for practical use, we let $k = 2$. Then the order of the method is 6. In the following, we only discuss the case $k = 2$. For convenience, we assume that $m = 1$, which can easily be generalized to m. When the Newton iteration is convergent, the matrix Q can be replaced by an approximation. In fact, we may use

$$J = \mathrm{diag}(J_{n+1}, J_{n+1}) \tag{4.4}$$

to approximate $\frac{\partial F(Y_m)}{\partial Y_m}$ and $\frac{\partial F(Y_{m+v})}{\partial Y_m}$. Then we have

$$Q \cong I - h(B - DA_*)J - h^2 DB_* J^2. \tag{4.5}$$

Let $k = 2, h = 0.1$. Some numerical results are given in Table 2; the number of iterations is two or three.

Example 1. $y' = 1/(1+x)^2 - 2y^2, y(0) = 0$.

Example 2. $y' = \frac{y}{4}(1 - \frac{y}{20}), y(0) = 1$.

Example 3. $y' = 1000x^3 - 1000y + 3x^2, y(0) = 0$.

Table 2

	Example 1		Example 2		Example 3	
x_n	y_n	error	y_n	error	y_n	error
0.5	.4	2.32E-08	1.125655	3.78E-08	.125	3.12E-10
1.0	.5	2.86E-09	1.266046	3.04E-08	1	1.27E-08
1.5	.4615385	1.01E-08	1.422627	5.01E-08	3.375002	4.62E-09
2.0	.4	1.73E-11	1.596923	1.97E-08	8.000003	2.14E-07
2.5	.3448276	2.23E-08	1.790516	8.76E-08	15.625	4.05E-08
3.0	.3	3.49E-08	2.00502	1.64E-08	26.99998	7.78E-07

Example 4. $\begin{cases} y' = 998y + 1998z, \\ z' = -999y - 1999z, y(0) = 1, z(0) = 0. \end{cases}$

This is a system of stiff equations. The eigenvalues are -1 and -1000, and the solution is

$$\begin{cases} y = 2e^{-x} - e^{-1000x}, \\ z = -e^{-x} + e^{-1000x} \end{cases}$$

Let $k = 2, h = 0.01$. The prescribed tolerance for iterations is $\varepsilon = 10^{-4}$. " Max error" denotes $\max\{|y_j - y(x_j)|, |z_j - z(x_j)|\}$. Then numerical results are given in Table 3.

Table 3

x	y	z	Max error
0.1	1.809528	−.9046904	1.47E-04
0.2	1.637466	−.8187357	4.99E-06
0.3	1.481635	−.7408174	1.48E-06
0.4	1.340636	−.6703158	4.53E-06
0.5	1.213059	−.6065297	2.07E-06

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