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# HIGHER ORDER FOLDS IN NONLINEAR PROBLEMS WITH SEVERAL PARAMETERS*

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**Abstract**

In this paper the results in [5] and [6] related to two-parameter nonlinear problems and computing the folds of degree 3 are generalized to any $n$-parameter nonlinear problems. Constructing a repeatedly extended system for an $n$-parameter nonlinear problem we prove that a fold of degree $n+1$ corresponds to a regular solution of its $n$-th extended system. Also, the equivalence between the $n$-th extended system and its reduced system is proved. Finally, some examples are computed.

## 1. Introduction

We consider an $n$-parameter nonlinear problem in the form

$$f(\lambda, \mu_1, \cdots, \mu_{n-1}, x) = 0 \tag{1.1}$$

where $\lambda, \mu_1, \cdots, \mu_{n-1} \in R, x \in X$, a Banach space, and $f$ is a $C^{n+1}$ mapping from $\underbrace{R \times \ldots \times R}_{n} \times X$ to $X$.

In many applications some loss of criticality in (1.1), which corresponds to a fold point of higher order at particular values $\lambda^*, \mu_1^*, \cdots, \mu_{n-1}^*$, is concerned. For example, the loss of criticality in the exothermic reaction described by a two-parameter nonlinear problem corresponds to two particular values $\lambda^*, \mu^*$ which are called the third degree fold point of $f(\lambda, \mu, x) = 0$ with respect to $\lambda$.

In the case $n = 2$, following the idea suggested in [2] and [4], Spence and Werner [5] proposed an "extended system" of the original problem, and proved that a third degree fold

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point of $f(\lambda,\mu,x)=0$ with respect to $\lambda$ corresponds to a second degree fold point of the extended system with respect to $\mu$. Yang and Keller [6] further developed a "double extended system" of $f(\lambda,\mu,x)=0$, and pointed out that a third degree fold point of $f(\lambda,\mu,x)=0$ with respect to $\lambda$ corresponds to a regular solution of the double extended system.

The outline of this paper is as follows. In Section 2, the one-parameter case is discussed and definitions of fold points are given. We introduce some special polynomial operators and discuss their properties. We prove a sufficient and necessary condition for a fold point of higher order.

In Section 3, we discuss the two-parameter case

$$f(\lambda,\mu,x)=0 \tag{1.2}$$

and consider the relation between (1.2) and its extended system. We generalize the results in [5] and prove that a fold point of degree $n+1$ of (1.2) with repect to $\lambda$ corresponds to a fold point of degree $n$ of its extended system with repect to $\mu$.

Section 4 contains the main results of the paper. We apply the idea of the extended system repeatedly for the $n$-parameter case

$$f(\lambda,\mu_1,\cdots,\mu_{n-1},x)=0. \tag{1.3}$$

We develop "the $n$-th extended system" of (1.3) and prove that a fold point of degree $n+1$ of (1.3) corresponds to a regular solution of its $n$-th extended system. A reduced system for the $n$-th extended system is introduced in order to compute the fold point of degree $n+1$ practically.

Section 5 contains two numerical examples in which there are folds of degree 3 and degree 4.

## 2. One–Parameter Case and Fold Points

We consider a one-parameter nonlinear problem in a Banach space $X$

$$f(\lambda,x)=0 \tag{2.1}$$

where $\lambda\in R, x\in X$ and $f$ is a $C^{n+1}$($n\geq 1$ is a suitable positive integer) mapping from $I\!R\times X$ to $X$.

The notations $f_\lambda(a), f_{\lambda\lambda}(a), f_x(a), f_{xx}(a), f_{\lambda x}(a), f_{xxx}(a),\cdots$ are used to denote the partial Frechet-derivatives of $f$ at $a=(\lambda,x)\in I\!R\times X$. We denote the dual pairing of $x\in X$ and $\psi\in X^*$ by $\psi x$ where $X^*$ is the conjugate space of $X$.

**Definition 2.1.** *A point $a_0 = (\lambda_0, x_0) \in I\!R \times X$ is a fold point of* (2.1) *with respect to $\lambda$ if*

$$f(a_0) = 0, \tag{2.2}$$

$$\text{Ker } f_x(a_0) \neq 0, \tag{2.3}$$

$$f_\lambda(a_0) \notin \text{Range } f_x(a_0). \tag{2.4}$$

**Definition 2.2.** *A fold point $a_0$ is a simple fold of* (2.1) *with respect to $\lambda$ if, in addition to (2.2)-(2.4)*

$$\dim \text{Ker } f_x(a_0) = \text{Codim Range } f_x(a_0) = 1. \tag{2.5}$$

*In this case there exist nontrivial $\phi_0 \in X$ and $\psi_0 \in X^*$ such that*

$$N \equiv \text{Ker } f_x(a_0) = \{\alpha\phi_0 \mid \alpha \in I\!R\}, \tag{2.6}$$

$$R \equiv \text{Range } f_x(a_0) = \{x \in X \mid \psi_0 x = 0\}. \tag{2.7}$$

As is well known, the zero set of $f(\lambda, x)$ near a simple fold $a_0$ is a smooth curve

$$\Gamma : f^{-1}(0) \cap U = \{(\lambda(s), x(s)) \Big| |s - s_0| \leq \delta\}$$

where $U$ is an open neighborhood of the simple fold point $a_0$, $\delta$ is a small positive and $\lambda(s), x(s)$ are $C^{n+1}$ mappings satisfying $\lambda(s_0) = \lambda_0, x(s_0) = x_0, |\lambda'(s)|^2 + \|x'(s)\|^2 > 0$ and $x(s) = (s - s_0)\phi_0 + v(s), v(s) \in V_0$ where $V_0$ is a complement of $N$, i.e. $X = N \oplus V_0$.

Along $\Gamma$ we have the identity $f(\lambda(s), x(s)) \equiv 0$. Instead of $f_\lambda(\lambda(s), x(s)), f_x(\lambda(s), x(s)), \cdots$ we shall write $f_\lambda, f_x, \cdots$ and denote $f_\lambda^0 = f_\lambda(\lambda(s_0), x(s_0)), f_x^0 = f_x(\lambda(s_0), x(s_0)), \cdots$. Differentiating $f(\lambda(s), x(s)) \equiv 0$ with respect to $s$ yields

$$f_\lambda\lambda'(s) + f_x x'(s) \equiv 0, \quad |s - s_0| \leq \delta. \tag{2.8}$$

Obviously

$$\lambda'(s_0) = 0, \tag{2.9}$$

$$x'(s_0) = \phi_0. \tag{2.10}$$

**Definition 2.3.** *A simple fold point $a_0 \in I\!R \times X$ is said to have degree $n+1$ if*

$$\lambda''(s_0) = \cdots = \lambda^{(n)}(s_0) = 0, \quad \lambda^{(n+1)}(s_0) \neq 0 \tag{2.11}$$

*where $\lambda^{(i)}(s_0)$ is the $i$-th derivative of $\lambda(s)$ with respect to $s$ at $s = s_0$. We also call it the $(n+1)$th degree fold.*

Before working out a sufficient and necessary condition for the $(n+1)$th degree fold we must introduce some special polynomial operators.

$$P_2(\phi_0; f) = f^0_{xx}\phi_0\phi_0, \tag{2.12}$$

$$P_3(\phi_0, \phi_1; f) = 3f^0_{xx}\phi_0\phi_1 + f^0_{xxx}\phi_0\phi_0\phi_0, \tag{2.13}$$

$$P_4(\phi_0, \phi_1, \phi_2; f) = 4f^0_{xx}\phi_0\phi_2 + 3f^0_{xx}\phi_1\phi_1 + 6f^{0'}_{xxx}\phi_0\phi_0\phi_1 + f^0_{xxxx}\phi_0\phi_0\phi_0\phi_0, \tag{2.14}$$

$$\begin{aligned} P_5(\phi_0, \phi_1, \phi_2, \phi_3; f) = {} & f^0_{xx}(5\phi_0\phi_3 + 10\phi_1\phi_2) + f^0_{xxx}(10\phi_0\phi_0\phi_2 + 15\phi_0\phi_1\phi_1) \\ & + f^0_{x^4}(10\phi_0^3\phi_1) + f^0_{x^5}\phi_0^5. \end{aligned} \tag{2.15}$$

The polynomial operator of degree $n$ $P_n(\phi_0, \phi_1, \cdots, \phi_{n-2}; f)$ in $n-1$ elements can be defined in the following way:

If $\lambda'(s_0) = 0, \cdots, \lambda^{(n)}(s_0) = 0, \lambda^{(n+1)}(s_0) \neq 0$ and $x'(s_0) = \phi_0, \cdots, x^n(s_0) = \phi_{n-1}, x^{(n+1)}(s_0) = \phi_n$, then

$$P_{n+1}(\phi_0, \cdots, \phi_{n-1}; f) = \frac{d^{n+1} f(\lambda(s), x(s))}{ds^{n+1}}\Big|_{s=s_0} - f^0_x\phi_n - f^0_\lambda\lambda^{(n+1)}(s_0). \tag{2.16}$$

**Theorem 2.1.** *$a_0 = (\lambda(s_0), x(s_0))$ is an $(n+1)$th degree fold (2.1) iff there exist $\phi_0 \in N, \phi_1, \cdots, \phi_{n-1} \in V_0$ such that*

$$\phi_0 = x'(s_0), \quad \phi_1 = x''(s_0), \cdots, \phi_{n-1} = x^{(n)}(s_0)$$

*which are uniquely determined by*

$$f^0_x\phi_{i-1} = -P_i(\phi_0, \cdots, \phi_{i-2}; f), \quad i = 2, \cdots, n \tag{2.17a}$$

*and $\phi_n = x^{(n+1)}(s_0)$ satisfies*

$$P_{n+1}(\phi_0, \cdots, \phi_{n-1}; f) + f^0_x\phi_n + f^0_\lambda\lambda^{(n+1)}(s_0) = 0. \tag{2.17b}$$

*Furthermore*

$$\begin{aligned} &\psi_0[P_i(\phi_0, \cdots, \phi_{i-2}; f)] = 0, \quad i = 2, \cdots, n, \\ &\psi_0[P_{n+1}(\phi_0, \cdots, \phi_{n-1}; f)] \neq 0. \end{aligned} \tag{2.18}$$

*Proof.* In the case $n = 2$, differentiating $f(\lambda(s), x(s)) = 0$ twice with respect to $s$ at $s = s_0$ yields

$$f^0_x x'(s_0) + f^0_\lambda\lambda'(s_0) = 0$$

and

$$f^0_{xx}x'(s_0)x'(s_0) + f^0_x x''(s_0) + 2f^0_{\lambda x}\lambda'(s_0)x'(s_0) + f^0_{\lambda\lambda}\lambda'(s_0)\lambda'(s_0) + f^0_\lambda\lambda''(s_0) = 0. \tag{2.19}$$

Substituting $\lambda'(s_0)=0, x'(s_0)=\phi_0$ and $\lambda''(s_0)=0$ into (2.19) yields

$$f_{xx}^0\phi_0\phi_0+f_x^0x''(s_0)=0. \tag{2.20}$$

$\phi_1=x''(s_0)\in V_0$ is uniquely given by (2.20). Multiplying $\psi_0$ on (2.20) yields $\psi_0[P_2(\phi_0;f)] = 0$. Differentiating (2.19) with respect to $s$ at $s=s_0$ and substituting $\lambda'(s_0)=0, \lambda''(s_0)= 0, x'(s_0)=\phi_0, x''(s_0)=\phi_1$ into it we obtain

$$P_3(\phi_0,\phi_1;f)+f_x^0x'''(s_0)+f_\lambda^0\lambda'''(s_0)=0. \tag{2.21}$$

Multiplying $\psi_0$ on (2.21) and noticing $\lambda'''(s_0)\neq 0$ we have

$$\psi_0[P_3(\phi_0,\phi_1;f)]\neq 0$$

and

$$\lambda'''(s_0)=\frac{-\psi_0[P_3(\phi_0,\phi_1;f)]}{\psi_0[f_\lambda^0]}.$$

So the theorem is true for $n=2$. According to mathematical induction we suppose that the theorem is true for $n=k$. First we prove the necessary condition for $n=k+1$. Differentiating $f(\lambda(s),x(s))=0$ from twice to $k+1$ times with respect to $s$ at $s=s_0$ and noticing $\lambda''(s_0)=0,\cdots,\lambda^{(k)}(s_0)=0, \lambda^{k+1}(s_0)\neq 0$ we obtain

$$\begin{aligned}
&P_2(\phi_0;f)+f_x^0\phi_1=0,\\
&\qquad\vdots\\
&P_k(\phi_0,\cdots,\phi_{k-2};f)+f_x^0\phi_{k-1}=0,\\
&P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)+f_x^0x^{(k+1)}(s_0)+f_\lambda^0\lambda^{(k+1)}(s_0)=0.
\end{aligned}$$

Multiplying each equation by $\psi_0$ and denoting $x^{(k+1)}(s_0)=\phi_k$ we get the necessary condition.

Secondly, differentiating $f(\lambda(s),x(s))=0$, $k$ times and $k+1$ times respectively and noticing the assumption of the induction for $n=k$, we obtain

$$P_k(\phi_0,\cdots,\phi_{k-2};f)+f_x^0\phi_{k-1}+f_\lambda^0\lambda^{(k)}(s_0)=0, \tag{2.22}$$

$$P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)+f_x^0\phi_k+f_\lambda^0\lambda^{(k+1)}(s_0)=0. \tag{2.23}$$

Multiplying (2.22) by $\psi_0$ yields

$$\lambda^k(s_0)=0$$

because $\psi_0[P_k(\phi_0,\cdots,\phi_{k-2};f)]=0$ and $\psi_0[f_\lambda^0]\neq 0$. Multiplying (2.23) by $\psi_0$ yields

$$\lambda^{(k+1)}(s_0)=\frac{-\psi_0[P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)]}{\psi_0[f_\lambda^0]}\neq 0$$

because $\psi_0[P_{k+1}(\phi_0, \cdots, \phi_{k-1}; f] \neq 0$ and $\psi_0[f^0] \neq 0$.

The sufficient condition is also proved. Q.E.D.

## 3. Two-Parameter Nonlinear Problems

The two-parameter nonlinear problem

$$f(\lambda, \mu, x) = 0 \quad , \tag{3.1}$$

is considered in this section, where $\lambda, \mu \in \mathbb{R}, x \in X$, a Banach space, and $f$ is a $C^{n+1}$ mapping from $\mathbb{R} \times \mathbb{R} \times X \to X$. Regarding $\lambda, \mu, x$ as functions of $s$ we consider nonlinear mapping $f(\lambda(s), \mu(s), x(s))$.

**Lemma 3.1.** *If*

$$\lambda'(s_0) = 0, \cdots, \lambda^{(n-1)}(s_0) = 0,$$
$$\mu'(s_0) = 0, \cdots, \mu^{(n-1)}(s_0) = 0,$$

*then*

$$\begin{aligned}&\frac{d^n}{ds^n}\Big(f_\lambda(\lambda(s), \mu(s), x(s))\lambda'(s) + f_\mu(\lambda(s), \mu(s), x(s))\mu'(s)\Big)\Big|_{s=s_0}\\ &= n\Big(f^0_{\lambda x}x'(s_0)\lambda^{(n)}(s_0) + f^0_{\mu x}x'(s_0)\mu^{(n)}(s_0)\Big) + f^0_\lambda\lambda^{(n+1)}(s_0) + f^0_\mu\mu^{(n+1)}(s_0).\end{aligned} \tag{3.2}$$

*Proof.* In the case $n = 2$, differentiating $f_\lambda\lambda' + f_\mu\mu'$ twice with respect to $s$ at $s = s_0$ we obtain

$$\frac{d^2}{ds^2}(f_\lambda\lambda' + f_\mu\mu')\Big|_{s=s_0} = 2\Big(f^0_{\lambda x}x'(s_0)\lambda''(s_0) + f^0_{\mu x}x'(s_0)\mu''(s_0)\Big) + f^0_\lambda\lambda'''(s_0) + f^0_\mu\mu'''(s_0).$$

So the theorem is true for $n = 2$.

The theorem is supposed to be true for $n = k - 1$ by induction.

$$\begin{aligned}\frac{d^{k-1}}{ds^{k-1}}(f_\lambda\lambda' + f_\mu\mu')\Big|_{s=s_0} &= (k-1)\Big(f^0_{\lambda x}x'(s_0)\lambda^{(k-1)}(s_0) + f^0_{\mu x}x'(s_0)\mu^{(k-1)}(s_0)\Big)\\ &\quad + f^0_\lambda\lambda^{(k)}(s_0) + f^0_\mu\mu^{(k)}(s_0)\end{aligned} \tag{3.3}$$

Differentiating (3.3) with respect to $s$ at $s = s_0$ and using $\lambda^{(k-1)}(s_0) = 0, \mu^{(k-1)}(s_0) = 0$ yields

$$\begin{aligned}\frac{d^k}{ds^k}(f_\lambda\lambda' + f_\mu\mu')\Big|_{s=s_0} &= (k-1)\Big(f^0_{\lambda x}x'(s_0)\lambda^{(k)}(s_0) + f^0_{\mu x}x'(s_0)\mu^{(k)}(s_0)\Big)\\ &\quad + f^0_{\lambda x}x'(s_0)\lambda^{(k)}(s_0) + f^0_{\mu x}x'(s_0)\mu^{(k)}(s_0) + f^0_\lambda\lambda^{(k+1)}(s_0) + f^0_\mu\mu^{(k+1)}(s_0)\\ &= k\Big(f^0_{\lambda x}x'(s_0)\lambda^{(k)}(s_0) + f^0_{\mu x}x'(s_0)\mu^{(k)}(s_0)\Big) + f^0_\lambda\lambda^{(k+1)}(s_0) + f^0_\mu\mu^{(k+1)}(s_0).\end{aligned}$$

Q.E.D.

Let

$$F(\mu,y)=(\ell\phi-1,f(\lambda,\mu,x),f_x\phi)^T=0, \tag{3.4}$$

where $y=(\lambda,x,\phi)^T\in Y=\mathbb{R}\times X\times X,\ell\in X^*$. $V$ is a complement of $N(F_y^0)$ in $Y$.

**Theorem 3.1.** *Suppose $F_\mu^0\notin$ Range $F_y^0$. An $(n+1)$th degree fold point $(\lambda_0,x_0,\phi_0)$ of (3.1) with respect to $\lambda$ corresponds to an nth fold point $(\mu_0,\lambda_0,x_0,\phi_0)$ of (3.4) with respect to $\mu$. And*

$$P_i(\Phi_0,\cdots,\Phi_{i-2};F)=\begin{pmatrix}0\\ P_i(\phi_0,\cdots,\phi_{i-2};f)\\ P_{i+1}(\phi_0,\cdots,\phi_{i-1};f)-f_{xx}^0\phi_0\phi_{i-1}\end{pmatrix}, \tag{3.5}$$
$$\forall\, i=2,\cdots n,$$

$$\Psi_0P_i(\Phi_0,\cdots,\Phi_{i-2};F)=\psi_0P_{i+1}(\phi_0,\cdots,\phi_{i-1};f),\quad \forall\, i=2,\cdots,n, \tag{3.6}$$

*where*

$$\Phi_i\equiv y^{(i+1)}(s_0)=(0,\phi_i,\phi_{i+1})^T,\quad \forall\, i=1,\cdots,n-2, \tag{3.7}$$

*$\Phi_i\in V$ is uniquely given by*

$$F_y^0\Phi_i=-P_{i+1}(\Phi_0,\cdots,\Phi_{i-1};F). \tag{3.8}$$

*Proof.* Theorem 3.1 in [5] shows the theorem is true for the case $n=2$.

Suppose the theorem is true for $n=k-1$ by induction and we consider the case $n=k$.

Differentiating $F(\mu,y)=0$ with respect to $s$ at $s=s_0$ up to $k-1$ times and noticing $\mu'(s_0)=0,\cdots,\mu^{(k-2)}(s_0)=0$ by the assumption of induction, we obtain

$$P_{k-1}(\Phi_0,\cdots,\Phi_{k-3};F)+F_y^0\Phi_{k-2}+F_\mu^0\mu^{(k-1)}(s_0)=0$$

where

$$\Phi_0=y'(s_0),\cdots,\Phi_{k-3}=y^{(k-2)}(s_0),\Phi_{k-2}=y^{(k-1)}(s_0).$$

Actually

$$\Phi_{i-1}=y^{(i)}(s_0)=\left(\lambda^{(i)}(s_0),x^{(i)}(s_0),x^{(i+1)}(s_0)\right)^T=(0,\phi_{i-1},\phi_i),\quad \forall i=1,\cdots,k-1,$$

because $(\lambda_0,\mu_0,x_0)$ is a $(k+1)$th degree fold point of (3.1) with respect to $\lambda$. By the assumption of induction we obtain

$$P_{k-1}(\Phi_0,\cdots,\Phi_{k-3};F)+F_y^0\Phi_{k-2}+F_\mu^0\mu^{(k-1)}(s_0)$$

$$=\begin{pmatrix}0\\ P_{k-1}(\phi_0,\cdots,\phi_{k-3};f)\\ P_k(\phi_0,\cdots,\phi_{k-2};f)-f_{xx}^0\phi_0\phi_{k-2}\end{pmatrix}+\begin{pmatrix}0 & 0 & \ell\\ f_\lambda^0 & f_x^0 & 0\\ f_{\lambda x}^0\phi_0 & f_{xx}^0\phi_0 & f_x^0\end{pmatrix}\begin{pmatrix}0\\ \phi_{k-2}\\ \phi_{k-1}\end{pmatrix}$$

$$+\begin{pmatrix}0\\ f_\mu^0\\ f_{\mu x}^0\phi_0\end{pmatrix}\mu^{(k-1)}(s_0)=0.$$

So

$$\mu^{(k-1)}(s_0)=0$$

and $\Phi_{k-2}\in V$ is also uniquely given by $(0,\phi_{k-2},\phi_{k-1})^T$ and satisfies (3.8).

Next we show that (3.5) and (3.6) are also true for $n=k$. By the definition of $P_n(\phi_0,\cdots,\phi_{n-2};f)$ and $\lambda^{(k)}(s_0)=0$ we have

$$P_k(\Phi_0,\cdots,\Phi_{k-2};F)=\frac{d^k}{ds^k}F(\mu(s_0),y(s_0))-F_y^0y^{(k)}(s_0)-F_\mu^0\mu^{(k)}(s_0)$$

$$=\begin{pmatrix}\ell x^{(k+1)}(s_0)\\ \dfrac{d^k}{ds^k}f(\lambda(s_0),\mu(s_0),x(s_0))\\ \dfrac{d^k}{ds^k}f_x(\lambda(s_0),\mu(s_0),x(s_0))\phi_0\end{pmatrix}$$

$$-\begin{pmatrix}0 & 0 & \ell\\ f_\lambda^0 & f_x^0 & 0\\ f_{\lambda x}^0\phi_0 & f_{xx}^0\phi_0 & f_x^0\end{pmatrix}\times\begin{pmatrix}\lambda^{(k)}(s_0)\\ x^{(k)}(s_0)\\ x^{(k+1)}(s_0)\end{pmatrix}-\begin{pmatrix}0\\ f_\mu^0\\ f_{\mu x}^0\phi_0\end{pmatrix}\mu^{(k)}(s_0)$$

$$=\begin{pmatrix}\ell x^{(k+1)}(s_0)-\ell x^{(k+1)}(s_0)\\ \dfrac{d^k}{ds^k}f(\lambda(s_0),\mu(s_0),x(s_0))-f_x^0x^{(k)}(s_0)-f_\mu^0\mu^{(k)}(s_0)\\ \dfrac{d^k}{ds^k}f_x(\lambda(s_0),\mu(s_0),x(s_0))\phi_0-\left(f_{xx}^0\phi_0x^{(k)}(s_0)+f_x^0x^{(k+1)}(s_0)+f_{\mu x}^0\phi_0\mu^{(k)}(s_0)\right)\end{pmatrix}.$$

Because $\mu'(s_0)=0,\cdots,\mu^{(k-1)}(s_0)=0$ and $\lambda^{(k)}(s_0)=0$ we obtain

$$\frac{d^k}{ds^k}f(\lambda(s_0),\mu(s_0),x(s_0))-f_x^0x^{(k)}(s_0)-f_\mu^0\mu^{(k)}(s_0)$$

$$=\frac{d^k}{ds^k}f(\lambda(s_0),\mu_0,x(s_0))-f_x^0x^{(k)}(s_0)-f_\lambda^0\lambda^{(k)}(s_0)$$

$$=P_k(\phi_0,\cdots,\phi_{k-2};f)$$

where $\mu_0$ in $\frac{d^k}{ds^k}f(\lambda(s_0),\mu_0,x(s_0))$ is fixed and is not regarded as a function of $s$.

Applying Lemma 3.1 to the third component yields

$$\frac{d^k}{ds^k}f_x(\lambda(s_0),\mu(s_0),x(s_0))\phi_0-\left(f_{xx}^0\phi_0x^{(k)}(s_0)+f_x^0x^{(k+1)}(s_0)+f_{\mu x}^0\phi_0\mu^{(k)}(s_0)\right)$$

$$=\frac{d^{k+1}}{ds^{k+1}}f(\lambda(s_0),\mu(s_0),x(s_0))-\frac{d^k}{ds^k}\Big(f_\lambda(\lambda(s_0),\mu(s_0),x(s_0))\lambda'(s_0)$$

$$+f_\mu(\lambda(s_0),\mu(s_0),x(s_0))\mu'(s_0)\Big)-(f_{xx}^0\phi_0x^{(k)}(s_0)+f_x^0x^{(k+1)}(s_0)+f_{\mu x}\phi_0\mu^{(k)}(s_0))$$

$$=\frac{d^{k+1}}{ds^{k+1}}f(\lambda(s_0),\mu(s_0),x(s_0))-\left(kf_{\mu x}^0\phi_0\mu^{(k)}(s_0)+f_\lambda^0\lambda^{(k+1)}(s_0)+f_\mu^0\mu^{(k+1)}(s_0)\right)$$

$$-(f_{xx}^0\phi_0x^{(k)}(s_0)+f_x^0x^{(k+1)}(s_0)+f_{\mu x}\phi_0\mu^{(k)}(s_0))$$

$$=\frac{d^{k+1}}{ds^{k+1}}f(\lambda(s_0)),\mu(s_0),x(s_0)-(k+1)f_{\mu x}^0\phi_0\mu^{(k)}(s_0)-f_\mu^0\mu^{(k+1)}(s_0)$$

$$-f_x^0x^{(k+1)}(s_0)-f_\lambda^0\lambda^{(k+1)}(s_0)-f_{xx}^0\phi_0x^{(k)}(s_0)$$

$$=\frac{d^{k+1}}{ds^{k+1}}f(\lambda(s_0),\mu_0,x(s_0)-f_x^0x^{(k+1)}(s_0))-f_\lambda^0\lambda^{(k+1)}(s_0)-f_{xx}^0\phi_0x^{(k)}(s_0)$$

$$=P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)-f_{xx}^0\phi_0\phi_{k-1}.$$

So

$$P_k(\Phi_0,\cdots,\Phi_{k-2};F)=\begin{pmatrix}0\\P_k(\phi_0,\cdots,\phi_{k-2};f)\\P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)-f_{xx}^0\phi_0\phi_{k-1}\end{pmatrix}$$

which is (3.5) for $n=k$.

Multiplying $P_k(\Phi_0,\cdots,\Phi_{k-2};F)$ by $\Psi_0=(0,\varsigma_0,\psi_0)$, where $\varsigma_0\in X^*$ is uniquely given by $\varsigma_0f_\lambda^0=-\psi_0f_{\lambda x}^0\phi_0,\varsigma_0f_x^0=-\psi_0f_{xx}^0\phi_0$ (See Theorem 2.1 in [5]), yields

$$\Psi_0P_k(\Phi_0,\cdots,\Phi_{k-2};F)=\varsigma_0P_k(\phi_0,\cdots,\phi_{k-2};f)+\psi_0(P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)-f_{xx}^0\phi_0\phi_{k-1}).$$

Noticing $\varsigma_0f_x^0=-\psi_0f_{xx}^0\phi_0$ and $f_x\phi_{k-1}=-P_k(\phi_0,\cdots,\phi_{k-2};f)$ we have proved (3.6) for $n=k$ and $\mu^{(k)}(s_0)\neq0$ from

$$\Psi_0P_k(\Phi_0,\cdots,\Phi_{k-2};F)=\psi_0P_{k+1}(\phi_0,\cdots,\phi_{k-1};f)\neq0.$$

Q. E. D.

**Note 3.1.** In the product space $Y = \mathbb{R}\times X\times X$ the kernel $N(F_y^0)$ is spanned by $\Phi_0 = (0,\phi_0,\phi_1)$. We normalise $\Phi_0$ by $L\Phi_0-1=0$ where $L\in \mathbb{R}\times X^*\times X^*$. $V=\{y\in Y|Ly=0\}$ is a suitable complement of $N(F_y^0)$. Choose $L=(0,\ell,0)$. $\Phi_i=(0,\phi_i,\phi_{i+1})^T, i=1,2,\cdots,n-2$, indeed belong to $V$ because $\phi_i\in V_0$.

## 4. An $N$-Parameter Nonlinear Problem and Its $n$th Extended System

We consider an $n$-parameter nonlinear problem

$$f(\lambda,\mu_1,\cdots,\mu_{n-1},x)=0 \tag{4.1}$$

where $\lambda,\mu_1,\cdots,\mu_{n-1}\in\mathbb{R}, x\in X$, a Banach space, and $f$ is a $C^{n+1}$ mapping from $\underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n}\times X$ to X.

Suppose $(\lambda_0,\mu_{1_0},\cdots,\mu_{n-1_0},x_0)$ is an $(n+1)$th degree fold point of (4.1) with respect to $\lambda$. Again we emphasise that $\mu_{1_0},\cdots,\mu_{n-1_0}$ are fixed and $\mu_1,\cdots,\mu_{n-1}$ are not regarded as functions of $s$. There are $\phi_0=x'(s_0),\cdots,\phi_{n-1}=x^{(n)}(s_0)$ satisfying (2.17) and (2.18). We introduce the first extended system of (4.1)

$$F^1(\mu_1,\cdots,\mu_{n-1},y_1)\equiv\begin{pmatrix}\ell_0\phi-1\\ f\\ f_x\phi\end{pmatrix}=0 \tag{4.2}$$

where $\ell_0\in X^*, y_1=(\lambda,x,\phi)^T\in Y_1=\mathbb{R}\times X\times X$, and $F^1$ is a $C^n$ mapping from $\underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n-1}\times Y_1$ to $Y_1$. By Theorem 3.1, $(\mu_{1_0},\cdots,\mu_{n-1_0},y_{1_0})$ is an $n$th degree fold point of (4.2) with respect to $\mu_1$ provided $(F^1_{\mu_1})^0\notin$ Range $(F^1_{y_1})^0$, where

$$y_{1_0}=(\lambda_0,x_0,\phi_0)^T. \tag{4.3}$$

$\mu_{2_0},\cdots,\mu_{n-1_0}$ are not fixed and $\mu_2,\cdots,\mu_{n-1}$ are not regarded as functions of $s$. There are $\Phi^1_0,\Phi^1_1,\cdots,\Phi^1_{n-2}$ satisfying (2.17) and (2.18) with corresponding spaces and notations. And

$$\Phi_i^{(1)}\equiv y_1^{(i+1)}(s_0)=(0,\phi_i,\phi_{i+1})^T,\quad i=0,\cdots,n-2. \tag{4.4}$$

The second extended system of (4.1)

$$F^2(\mu_2,\cdots,\mu_{n-1},y_2)\equiv\begin{pmatrix}\ell_1\Phi^1-1\\ F^1\\ F^1_{y_1}\Phi^1\end{pmatrix}=0 \tag{4.5}$$

is the first extended system of (4.2). $l_1\in Y^*$ is chosen $(0,\ell_0,0)$ later on. $y_2=(\mu_1,y_1,\Phi^1)^T\in Y_2=\mathbb{R}\times Y_1\times Y_1$ and $F^2$ is a $C^{n-1}$ mapping from $\underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n-2}\times Y_2$ to $Y_2$. Again applying

Theorem 3.1 yields that $(\mu_{2_0},\cdots,\mu_{n-1_0},y_{2_0})$ is an $(n-1)$th degree fold point of (4.5) with respect to $\mu_2$ provided $(F^2_{\mu_2})^0 \notin \mathrm{Range}(F^2_{y_2})^0$, where

$$y_{2_0}=(\mu_{1_0},y_{1_0},\Phi_0^1), \tag{4.6}$$

$\mu_{3_0},\cdots,\mu_{n-1_0}$ are fixed and $\mu_3,\cdots,\mu_{n-1}$ are not regarded as functions of $s$. There are $\Phi_0^2,\Phi_1^2,\cdots,\Phi_{n-3}^2$ satisfying (2.17) and (2.18) with corresponding spaces and notations. And

$$\Phi_i^2\equiv y_2^{(i+1)}(s_0)=(0,\Phi_i^1,\Phi_{i+1}^1)^T,\quad i=0,\cdots,n-3.$$

Likewise, the third extended system of (4.1)

$$F^3(\mu_3,\cdots,\mu_{n-1},y_3)\equiv\begin{pmatrix}\ell_2\Phi^2-1\\ F^2\\ F^2_{y_2}\Phi^2\end{pmatrix}=0 \tag{4.8}$$

is the first extended system of (4.5). $\ell_2\in Y_2^*$ is chosen $(0,\ell_1,0)$ later on . $y_3=(\mu_2,y_2,\Phi^2)^T\in Y_3=I\!R\times Y_2\times Y_2$ and $F^3$ is a $C^{n-2}$ mapping from $\underbrace{I\!R\times\cdots\times I\!R}_{n-3}\times Y_3$ to $Y_3$.

Applying Theorem 3.1 again yields that $(\mu_{3_0},\cdots,\mu_{n-1_0},y_{3_0})$ is an $(n-2)$th degree fold point of (4.8) with respect to $\mu_3$ provided $(F^3_{\mu_3})^0\notin\mathrm{Range}(F^3_{y_3})^0$, where

$$y_{3_0}=(\mu_{2_0},y_{2_0},\Phi_0^2). \tag{4.9}$$

Again $\mu_{4_0},\cdots,\mu_{n-1_0}$ are fixed and $\mu_4,\cdots,\mu_{n-1}$ are not regarded as functions of $s$. There are $\Phi_0^3,\Phi_1^3,\cdots,\Phi_{n-4}^3$ satisfying (2.17) and (2.18) with corresponding spaces and notations. And

$$\Phi_i^3\equiv y_3^{(i+1)}(s_0)=(0,\Phi_i^2,\Phi_{i+1}^2),\quad i=0,\cdots,n-4. \tag{4.10}$$

We define the $(n-1)$th extended system of (4.1) recursively

$$F^{n-1}(\mu_{n-1},y_{n-1})\equiv\begin{pmatrix}\ell_{n-2}\Phi^{n-2}-1\\ F^{n-2}\\ F^{n-1}_{y_{n-2}}\Phi^{n-2}\end{pmatrix}=0 \tag{4.11}$$

which is the first extended system of $F^{n-2}(\mu_{n-2},\mu_{n-1},y_{n-2})=0$. $\ell_{n-2}\in Y_{n-2}^*$ is chosen $(0,\ell_{n-3},0)$ later on. $y_{n-1}=(\mu_{n-2},y_{n-2},\Phi^{n-2})^T\in Y_{n-1}=I\!R\times Y_{n-2}\times Y_{n-2}$ and $F^{n-1}$ is a $C^2$ mapping from $I\!R\times Y_{n-1}$ to $Y_{n-1}$. Applying Theorem 3.1 again yields that $(\mu_{n-1_0},y_{n-1_0})$ is a second degree fold point of (4.11) with respect to $\mu_{n-1}$ provided $(F^{n-1}_{\mu_{n-1}})^0\notin\mathrm{Range}(F^{n-1}_{y_{n-1}})^0$, where

$$y_{n-1_0}=(\mu_{n-2_0},y_{n-2_0},\Phi_0^{n-2}). \tag{4.12}$$

There is

$$\Phi_0^{n-1}\equiv y'_{n-1}(s_0)=(0,\Phi_0^{n-2},\Phi_1^{n-2}). \tag{4.13}$$

satisfying

$$\Psi_0^{n-1}\left(F^{n-1}_{y_{n-1}y_{n-1}}\right)^0\Phi_0^{n-1}\Phi_0^{n-1}\neq 0$$

where $\Psi_0^{n-1}\in Y_{n-1}^*$ is given by Range $(F^{n-1}_{y_{n-1}})^0=\{y_{n-1}\in Y_{n-1}|\Psi_0^{n-1}y_{n-1}=0\}$.

Finally, the $n$th extended system of (4.1) is defined recursively by

$$F^n(y_n)\equiv\begin{pmatrix}\ell_{n-1}\Phi^{n-1}-1\\ F^{n-1}\\ F^{n-1}_{y_{n-1}}\Phi^{n-1}\end{pmatrix}=0 \tag{4.14}$$

where $\ell_{n-1}\in Y_{n-2}^*$ is chosen $(0,\ell_{n-2},0)$ later on, $y_n=(\mu_{n-1},y_{n-1},\Phi^{n-1})^T\in Y_n=\mathbb{R}\times Y_{n-1}\times Y_{n-1}$ and $F^n$ is a $C^1$ mapping from $Y_n$ to $Y_n$. By Theorem 3.1,

$$y_{n_0}=(\mu_{n-1_0},y_{n-1_0},\Phi_0^{n-1}) \tag{4.15}$$

is a regular solution of (4.14).

In summary we obtain

**Theorem 4.1.** *Suppose* $(F^i_{\mu_i})^0\notin$ Range $(F^i_{y_i})^0, i=1,\cdots,n-1$. *An* $(n+1)$*th fold point* $(\lambda_0,\mu_{1_0},\cdots,\mu_{n-1_0},x_0)$ *of* (4.1) *with respect to* $\lambda$ *corresponds to a regular solution* $y_{n_0}$ *of the nth extended system* (4.14) *of* (4.1) *where* $y_{n_0}$ *is given recursively by* (4.15), (4.13), (4.12), (4.10), (4.9), (4.7), (4.6), (4.4) *and* (4.3).

Because $F^n(y_n)=0$ is no longer a singular system, Newton's method can be used for solving (4.14). But the $n$th extended system it too complicated to solve, and should be simplified.

In the extending procedure we can define polynomial operators

$$Q_1^i\equiv Q_1(\Phi_0^i;F^i)=(F^i_{y_i})^0\Phi_0^i,\quad i=0,\cdots,n-1, \tag{4.16a}$$

$$Q_j^i\equiv Q_j(\Phi_0^i,\cdots,\Phi_{j-1}^i;F^i)=P_j(\Phi_0^i,\cdots,\Phi_{j-2}^i;F)+(F^i_{y_i})^0\Phi_{j-1}^i,$$
$$j=2,\cdots,n;i=0,\cdots,n-j, \tag{4.16b}$$

where $F^0\equiv f,\Phi_k^0\equiv\phi_k,k=0,\cdots,n-1$, and $(F^i_{y_i})^0$ means $F^i_{y_i}$ is evaluated at the corresponding fold point. For simplicity we drop the symbol $(\ )^0$ below.

**Lemma 4.1.** *Under the conditions of* Theorem 4.1 *there exist*

$$Q_j(\Phi_0^i,\cdots,\Phi_{j-1}^i;F^i)=\begin{pmatrix}\ell_{i-1}\Phi_j^{i-1}\\ Q_j(\Phi_0^{i-1},\cdots,\Phi_{j-1}^{i-1};F^{i-1})\\ Q_{j+1}(\Phi_0^{i-1},\cdots,\Phi_j^{i-1};F^{i-1})\end{pmatrix}, \tag{4.17}$$

$$j=1,2,\cdots,n;i=1,2,\cdots,n-j.$$

*Proof.*

$$F^i_{y_i}\Phi_0^i=\begin{pmatrix}0&0&\ell_{i-1}\\ F^{i-1}_{\mu_{i-1}}&F^{i-1}_{y_{i-1}}&0\\ F^{i-1}_{\mu_{i-1}y_{i-1}}\Phi_0^{i-1}&F^{i-1}_{y_{i-1}y_{i-1}}\Phi_0^{i-1}&F^{i-1}_{y_{i-1}}\end{pmatrix}\begin{pmatrix}0\\ \Phi_0^{i-1}\\ \Phi_1^{i-1}\end{pmatrix},\quad\forall i=1,\cdots,n-1.$$

So

$$Q_1(\Phi_0^i; F^i) = \begin{pmatrix} \ell_{i-1}\Phi_1^{i-1} \\ Q_1(\Phi_0^{i-1}; F^{i-1}) \\ Q_2(\Phi_0^{i-1}, \Phi_1^{i-1}; F^{i-1}) \end{pmatrix}, \quad \forall\, i = 1, \cdots n-1,$$

which is (4.17) for the case $j = 1$.

Generally we have

$$F_{y_i}^i \Phi_{j-1}^i = \begin{pmatrix} 0 & 0 & \ell_{i-1} \\ F_{\mu_{i-1}}^{i-1} & F_{y_{i-1}}^{i-1} & 0 \\ F_{\mu_{i-1}y_{i-1}}^{i-1}\Phi_0^{i-1} & F_{y_{i-1}y_{i-1}}^{i-1}\Phi_0^{i-1} & F_{y_{i-1}}^{i-1} \end{pmatrix} \begin{pmatrix} 0 \\ \Phi_{j-1}^{i-1} \\ \Phi_j^{i-1} \end{pmatrix} \tag{4.18}$$

and

$$P_j(\Phi_0^i, \cdots, \Phi_{j-2}^j; F^i) = \begin{pmatrix} 0 \\ P_j(\Phi_0^{i-1}, \cdots, \Phi_{j-2}^{i-1}; F^{i-1}) \\ P_{j+1}(\Phi_0^{i+1}, \cdots, \Phi_{j-1}^{i-1}; F^{i-1}) - F_{y_{i-1}y_{i-1}}^{i-1}\Phi_0^{i-1}\Phi_{j-1}^{i-1} \end{pmatrix} \tag{4.19}$$

from (3.5). Adding (4.18) and (4.19) yields (4.17). Q.E.D.

**Theorem 4.2.** *The nth extended system of the n-parameter nonlinear problem $f(\lambda, \mu_1, \cdots, \mu_{n-1}, x) = 0$, which satisfies the assumptions in Theorem 4.1 , is equivalent to a reduced system of order n*

$$\begin{pmatrix} \ell_0\phi_0 - 1 \\ f \\ f_x\phi_0 \\ \ell_0\phi_1 \\ f_x\phi_1 + f_{xx}\phi_0\phi_0 \\ \ell_0\phi_2 \\ f_x\phi_2 + 3f_{xx}\phi_0\phi_1 + f_{xxx}\phi_0\phi_0\phi_0 \\ \vdots \\ \ell_0\phi_{n-1} \\ Q_n(\phi_0, \cdots, \phi_{n-1}; f) \end{pmatrix} = 0. \tag{4.20}$$

*Proof.* Generally we take

$$\ell_i = (0, \ell_{i-1}, 0), \qquad i = 1, \cdots, n-1.$$

At the regular solution $y_{n_0}$ of the nth extended system (4.14) of (4.1), applying repeatedly

Lemma 4.1 and (4.4), (4.7), (4.10), (4.13) and using the abbreviation $Q_j^i$ we have

$$
\begin{aligned}
F^n(y_{n_0}) &= \big(\ell_{n-1}\Phi_0^{n-1}-1, F^{n-1}, F^{n-1}_{y_{n-1}}\Phi_0^{n-1}\big)^T \\
&= \big(\ell_{n-2}\Phi_0^{n-2}-1, (\ell_{n-2}\Phi_0^{n-2}-1, F^{n-2}, Q_1^{n-2}), (\ell_{n-2}\Phi_1^{n-2}, Q_1^{n-2}, Q_2^{n-2})\big)^T \\
&= \big(\ell_{n-3}\Phi_0^{n-3}-1, (\ell_{n-3}\Phi_0^{n-3}-1, (\ell_{n-3}\Phi_0^{n-3}-1, F^{n-3}, Q_1^{n-3}), \\
&\quad (\ell_{n-3}\Phi_1^{n-3}, Q_1^{n-3}Q_2^{n-3})), (\ell_{n-3}\Phi_1^{n-3}, (\ell_{n-3}\Phi_1^{n-3}, Q_1^{n-3}, Q_2^{n-3}), \\
&\quad (\ell_{n-3}\Phi_2^{n-3}, Q_2^{n-3}, Q_3^{n-3}))\big)^T = \cdots\cdots\cdots \\
&= \Bigg(\ell_0\phi_0-1, \Bigg(\ell_0\phi_1-1, \Big(\ell_0\phi_0-1, (\ell_0\phi_0-1, \cdots, (\ell_0\phi_0-1, \\
&\quad (\ell_0\phi_0-1, f, Q_1^0), (\ell_0\phi_1, Q_1^0, Q_2^0)), \\
&\quad \cdots(\ell_0\phi_{n-4}, (\ell_0\phi_{n-4}, Q_{n-4}^0, Q_{n-3}^0), (\ell_0\phi_{n-3}, Q_{n-3}^0, Q_{n-2}^0))\cdots)\Big), \\
&\quad \Big(\ell_0\phi_1, (\ell_0\phi_1, \cdots, (\ell_0\phi_1, (\ell_0\phi_1, Q_1^0, Q_2^0), (\ell_0\phi_2, Q_2^0, Q_3^0)), \cdots \\
&\quad \cdots(\ell_0\phi_{n-3}, (\ell_0\phi_{n-3}, Q_{n-3}^0, Q_{n-2}^0), (\ell_0\phi_{n-2}, Q_{n-2}^0, Q_{n-1}^0))\cdots)\Big)\Bigg), \\
&\quad \Bigg(\ell_0\phi_1, \Big(\ell_0\phi_1, (\ell_0\phi_1, \cdots, (\ell_0\phi_1, (\ell_0\phi_1, Q_1^0, Q_2^0), (\ell_0\phi_2, Q_2^0, Q_3^0)), \\
&\quad \cdots(\ell_0\phi_{n-3}, (\ell_0\phi_{n-3}, Q_{n-3}^0, Q_{n-2}^0), (\ell_0\phi_{n-2}, Q_{n-2}^0, Q_{n-1}^0)), \cdots)\Big), \\
&\quad \Big(\ell_0\phi_2, (\ell_0\phi_0, \cdots, (\ell_0\phi_2, (\ell_0\phi_2, Q_2^0, Q_3^0), (\ell_0\phi_2, Q_3^0, Q_4^0))\cdots \\
&\quad \cdots(\ell_0\phi_{n-2}, \ell_0\phi_{n-2}, Q_{n-2}^0, Q_{n-1}^0), (\ell_0\phi_{n-1}, Q_{n-1}^0, Q_n^0))\cdots)\Big)\Bigg)\Bigg)^T.
\end{aligned}
\tag{4.21}
$$

By deleting the same terms, the right side of (4.21) is just the evaluation of the left side of (4.20) at $(\lambda_0, \mu_{1_0}, \cdots, \mu_{n-1_0}, x_0, \phi_0, \phi_1, \cdots, \phi_{n-1})$.

On the other hand, if there are $\lambda_0, \mu_{1_0}, \cdots, \mu_{n-1_0}, x_0, \phi_0, \cdots, \phi_{n-1}$ satisfying (4.20), then inversing the procedure in (4.21) we obtain

$$
\begin{aligned}
y_{n_0} = &\Bigg(\mu_{n-1_0}, \Bigg(\mu_{n-2_0}, \Big(\mu_{n-3_0}, (\mu_{n-4_0}, \cdots, (\mu_{1_0}, (\lambda_0, x_0, \phi_0), (0, \phi_0, \phi_1)), \\
&\cdots\cdots, (0, (0, \phi_{n-5}, \phi_{n-4}), (0, \phi_{n-4}, \phi_{n-3}))\cdots\Big)\Big), \\
&\Big(0, \Big(0, \cdots, (0, (0, \phi_0, \phi_1), (0, \phi_1, \phi_2)), \cdots, (0, (0, \phi_{n-4}, \phi_{n-3}), (0, \phi_{n-3}, \phi_{n-2}))\cdots\Big)\Big)\Bigg), \\
&\Bigg(0, \Big(0, \Big(0, \cdots, (0, (0, \phi_0, \phi_1), (0, \phi_1, \phi_2)), \cdots\cdots, \\
&(0, (0, \phi_{n-4}, \phi_{n-3}), (0, \phi_{n-3}, \phi_{n-2})\Big)\cdots\Big)\Big), \Big(0, \Big(0, \cdots, (0, (0, \phi_1, \phi_2), (0, \phi_2, \phi_3)), \\
&\cdots, (0, (0, \phi_{n-3}, \phi_{n-2}), (0, \phi_{n-2}, \phi_{n-1}))\cdots)\Big)\Big)\Bigg)\Bigg)
\end{aligned}
\tag{4.22}
$$

which is a regular solution of (4.20). Q.E.D.

**Note 4.1.** If the Banach space is a finite dimensional space (its dimensions are $m$), then the dimensions of the $n$th extended system (4.14) of (4.1) are $2^n(m+1)-1$. But the dimensions of the reduced system (4.20) of order $n$ are greatly decreased to $(n+1)(m+1)-1$.

Because of the regularity of the reduced system (4.20) at the $(n+1)$th fold point, we can use Newton's method to solve it. The similar algorithm with [6] can reduce a problem of solving one linear system of $(n+1)(m+1)-1$ dimensions in Newton's iteration to a problem of solving $n+1$ linear systems of $m$ dimensions with the same matrix.

## 5. Numerical examples

*Example* 1. An exothermic chemical reaction in an infinite slab can be described by the boundary value problem

$$\begin{aligned} &f(\lambda,\mu,x)=x''+\lambda\exp(x/(1+\mu x))=0,\\ &x(0)=x(1)=0 \end{aligned} \tag{5.1}$$

where $x$ is the dimensionless temperature, $\lambda$ is a rate parameter and $\mu$ is related to the activation energy.

In [6], we computed a fold of degree 3 in this two-parameter nonlinear problem. It is

$$\lambda_0=5.22959,\quad \mu_0=0.24578,\quad x(\frac{1}{2})=4.89655$$

which corresponds to the loss of criticality in the exothermic reaction.

*Example* 2. An axial dispersion problem in a tubular non-adiabatic reaction with the first-order exothermic reaction can be characterized by two partial differential equations (see [7] for details)

$$\begin{aligned} \frac{\partial y}{\partial t}&=\frac{1}{Pe_y}\frac{\partial^2 y}{\partial z^2}-\frac{\partial y}{\partial z}+Da(1-y)\exp(\theta/(1+\theta/\gamma)),\\ \frac{\partial \theta}{\partial t}&=\frac{1}{Pe_\theta}\frac{\partial^2 \theta}{\partial z^2}-\frac{\partial \theta}{\partial z}+BDa(1-y)\exp(\theta/(1+\theta/\gamma))-\beta(\theta-\theta_c) \end{aligned} \tag{5.2}$$

with the boundary conditions

$$\begin{aligned} z=0:&\qquad Pe_y\times y=\frac{\partial y}{\partial z},\qquad Pe_\theta\times\theta=\frac{\partial\theta}{\partial z},\\ z=1:&\qquad \frac{\partial y}{\partial z}=\frac{\partial\theta}{\partial z}=0. \end{aligned} \tag{5.3}$$

Here $y$ is dimensionless conversion, $y\in[0,1)$; $\theta$ is the dimensionless temperature, $Da$ is Damköhler number, $\gamma$ is dimensionless activation energy, $Pe_y$ is the Peclet number for axial mass transport, $Pe_\theta$ is the Péclet number for axial heat transport, $B$ is the dimensionless parameter of heat evolution, and $\theta_c$ is the dimensionless cooling temperature. All parameters are positive; $\theta_c$ can also be negative.

We are concerned with the multiple steady state solution of (5.2), (5.3). Instead of (5.2) we consider

$$\begin{aligned} &\frac{1}{Pe_y}\frac{d^2y}{dz^2} - \frac{dy}{dz} + Da(1-y)\exp(\theta/(1+\theta/\gamma)) = 0, \\ &\frac{1}{Pe_\theta}\frac{d^2\theta}{dz^2} - \frac{d\theta}{dz} + BDa(1-y)\exp(\theta/(1+\theta/\gamma)) - \beta(\theta-\theta_c) = 0. \end{aligned} \tag{5.2)'$$

To discrete (5.2)′ we use the central differences on the mesh points $z = jh, j = 0, \cdots, N$, where $Nh = 1$.We choose $N = 40$.

Fixing $\gamma = 20, Pe_y = 10, Pe_\theta = 5, B = 15$, we have a nonlinear problem with 3 parameters $Da, \beta, \theta_c$. The following is the family of the folds of degree 3 along with $\theta_c$.

| $\theta_c$ | $Da$ | $\beta$ | $y(0)$ | $y(\frac{1}{2})$ | $y(1)$ | $\theta(0)$ | $\theta(\frac{1}{2})$ | $\theta(1)$ |
|---|---|---|---|---|---|---|---|---|
| −0.1 | .18408 | 3.59639 | .037 | .352 | .688 | .520 | 2.156 | 2.831 |
| 0 | .16982 | 3.54926 | .034 | .316 | .655 | .501 | 2.045 | 2.927 |
| 2 | .04320 | 3.09272 | .015 | .195 | .608 | .865 | 3.091 | 5.405 |
| 4 | .01386 | 2.53719 | .008 | .148 | .591 | 1.269 | 4.306 | 7.616 |
| 6 | .00578 | 2.04955 | .006 | .122 | .580 | 1.586 | 5.352 | 9.528 |

The fold of degree 3 at $\theta_c = -0.1$ can be chosen as the initial guess for the fold of degree 4. Finally we compute the fold of degree 4 as follows:

$$\begin{aligned} &Da = 0.18434, \quad \beta = 3.59766, \quad \theta_c = -0.10147, \quad y(0) = 0.038, \\ &y(\tfrac{1}{2}) = 0.358, \quad y(1) = 0.694, \quad \theta(0) = 0.525, \quad \theta(\tfrac{1}{2}) = 2.189, \\ &\theta(1) = 2.845 \end{aligned}$$

which corresponds to the organizing center of the 3-parameter nonlinear problem with fixed $\gamma = 20, Pe_y = 10, Pe_\theta = 5, B = 15$ in (5.2)′, (5.3).

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