INTERVAL ITERATIVE METHODS UNDER PARTIAL ORDERING (I)*

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Abstract

Many types of nonlinear systems, which can be solved by ordered iterative methods, are discussed in unified form in the present paper. Under different initial conditions, some generalized ordered iterative methods are given, and the existence and uniqueness of the solution and the convergence of the methods are proved.

§ 1. Introduction

In this paper we consider nonlinear systems

$$\varphi(x)=x, \quad x\in R^n. \tag{1.1}$$

The partial ordering relation in R^n will be denoted, as usual, by "$\leqslant$", that is

$$x\leqslant y\leftrightarrow x_i\leqslant y_i \quad i=1, 2, \cdots, n,$$
$$x<y\leftrightarrow x_i<y_i \quad i=1, 2, \cdots, n.$$

For

$$\varphi(x)=g(x)+h(x)+c \tag{1.2}$$

where g, h: $R^n\to R^n$ are isotone and antitone mappings respectively, the problem has been solved quite satisfactorily. The purpose of this paper is to generalize the results in [1]—[4].

Suppose there are f_i: $R^{r_i}\times R^{s_i}\to R$, such that

$$\varphi_i(x)=f_i(A_ix, B_ix), \quad i=1, 2, \cdots, n \tag{1.3}$$

where $A_i\in R^{r_i\times n}$, $B_i\in R^{s_i\times n}$, $0\leqslant r_i$, $s_i\leqslant n$, $f_i(A_ix, B_iy)$ is isotone in x and antitone in y when they are comparable, that is, as $x\leqslant x'$, $y\geqslant y'$, $x\leqslant y$ or $y\leqslant x$, $x'\leqslant y'$ or $y'\leqslant x'$, we have

$$f_i(A_ix, B_iy)\leqslant f_i(A_ix', B_iy'), \quad i=1, 2, \cdots, n.$$

Example 1. For (1.2) we let

$$f(x, y)=g(x)+h(y)+c,$$
$$\varphi_i(x)=f_i(A_ix, B_ix)=g_i(A_ix)+h_i(B_ix)+c_i,$$

$A_i=B_i=I\in R^{n\times n}$, $i=1, 2, \cdots, n$.

Example 2[1]. For φ being diagonally isotone and off-diagonally antitone, we let

$$f_i(A_ix, B_iy)=\varphi_i(y+(x_i-y_i)e_i),$$
$$\varphi_i(x)=f_i(A_ix, B_ix),$$

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$A_i = e_i^T \in R^{1\times n}$, $B_i = [e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_n]^T \in R^{(n-1)\times n}$, $i=1, 2, \cdots, n$, where $e_i = [0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0]^T \in R^n$.

Example 3[5]. $\varphi(x) = x - q'(x)^{-1} q(x)$, where q is order convex on a convex set $D \subseteq R^n$ i. e.

$$q(\lambda x + (1-\lambda)y) \leqslant \lambda q(x) + (1-\lambda) q(y)$$

whenever $x, y \in D$, $x \leqslant y$ or $y \leqslant x$ and $\lambda \in (0, 1)$. And if q is G-differentiable, $q'(x) \geqslant 0$, $q'(x)$ is isotone and $q(x) \geqslant 0$, then

$$\varphi_i(x) = f_i(A_i x), \quad i=1, 2, \cdots, n,$$

$A_i = I$, $s_i = 0$. From

$$q'(\underline{x})(\bar{x} - \underline{x}) \leqslant q(\bar{x}) - q(\underline{x}) \leqslant q'(\bar{x})(\bar{x} - \underline{x}), \quad \underline{x} \leqslant \bar{x}$$

we can prove that φ is isotone.

Most of the functions discussed in [1] (13.2—13.5) can be written in form of (1.3).

For simplicity, we suppose $A = A_i$, $B = B_i$, $i=1, 2, \cdots, n$, and consider

$$\varphi(x) = f(Ax, Bx) = x. \tag{1.4}$$

Clearly (1.4) is equivalent to (1.1). For other case, we can get similar results.

We define an n-dimensional interval vector

$$[\underline{x}, \bar{x}] = \{u \mid \underline{x} \leqslant u \leqslant \bar{x}\}$$

as an order interval, $\underline{x}, \bar{x} \in R^n$, and define.

$$N = \{1, 2, \cdots, n\},$$
$$[\underline{x}, \bar{x}] \subseteq [\underline{y}, \bar{y}] \leftrightarrow \underline{y} \leqslant \underline{x} \leqslant \bar{x} \leqslant \bar{y},$$
$$[\underline{x}, \bar{x}]_i \subset [\underline{y}, \bar{y}]_i \leftrightarrow \underline{y}_i \leqslant \underline{x}_i \leqslant \bar{x}_i \leqslant \bar{y}_i \quad \text{and} \quad \bar{y}_i - \underline{y}_i > \bar{x}_i - \underline{x}_i,$$
$$W[\underline{x}, \bar{x}] = (\bar{x}_1 - \underline{x}_1, \cdots, \bar{x}_n - \underline{x}_n),$$
$$|x| = (|x_1|, |x_2|, \cdots, |x_n|)^T.$$

We will use the following lemmas.

Lemma 1. *Let $A \geqslant 0$ be an $n\times n$ matrix, and $\rho(A)$ be the spectral radius of the matrix A. Then*

(1) *A has a nonnegative real eigenvalue equal to its spectral radius.*

(2) *To $\rho(A)$, there is a corresponding eigenvector $x \geqslant 0$.*

(3) *$\rho(A)$ does not decrease when any entry of A is increased.*

(4) *$\alpha > \rho(A)$, if and only if $\alpha I - A$ is nonsingular and $(\alpha I - A)^{-1} \geqslant 0$.*

(5) *If A is an irreducible matrix, $\rho(A)$ increases when any entry of A increases.*

This lemma is a conclusion of the theorems about nonnegative matrices developed by Varge[6].

Lemma 2. *Let $B = I - w(\alpha I - A)$, $A \geqslant 0$, $\rho(A) < \alpha$, and $0 < w \leqslant \min_i \{1/(1-a_{ii})\}$, $a_{ii} = e_i^T A e_i$. Then $B \geqslant 0$ and $\rho(B) < 1$.*

Proof. From Lemma 1(4), $\alpha I - A$ is nonsingular and $(\alpha I - A)^{-1} \geqslant 0$. Therefore $\alpha - a_{ii} > 0$, $i=1, 2, \cdots, n$, and $B = I - w(\alpha I - A) \geqslant 0$. Since $I - B = w(\alpha I - A)$, $(I-B)^{-1} \geqslant 0$. Let λ be an eigenvalue of B, and x be an eigenvector of B corresponding to the eigenvalue λ. Then

$$|\lambda| |x| = |\lambda x| = |Bx| \leqslant |B| |x| = B|x|$$

$$(1-|\lambda|)|x| \geq (I-B)|x|$$
$$(1-|\lambda|)(I-B)^{-1}|x| \geq |x|$$

Since $|x| \neq 0$, $1>|\lambda|$ and $\rho(B)<1$.

Lemma 3. *Let $A\geq 0$ be an $n\times n$ matrix. If there exist $x\in R^n$, $x>0$, $\alpha>0$ such that*

$$Ax<\alpha x,$$

then $\rho(A)<\alpha$.

Proof. Since $A\geq 0$, $\rho(A)$ is an eigenvalue of A and there is an eigenvector $y\geq 0$ of A corresponding to $\rho(A)$.

Let

$$u=\min_{y_i\neq 0} x_i/y_i = x_{i_0}/y_{i_0}.$$

Then

$$x-uy\geq 0, \quad x_{i_0}-uy_{i_0}=0.$$

From $A\geq 0$, we have

$$A(x-uy)\geq 0,$$
$$\alpha x>Ax\geq uAy=u\rho(A)y,$$
$$\alpha x_{i_0}>\rho(A)uy_{i_0}=\rho(A)x_{i_0}.$$

Therefore $\rho(A)<\alpha$.

§ 2. Algorithm, Existence and Uniqueness, and Convergence

Suppose

$$F[\underline{x}, \bar{x}]=[f(A\underline{x}, B\bar{x}), f(A\bar{x}, B\underline{x})].$$

We consider five initial conditions:

1. $F[\underline{x}^0, \bar{x}^0]\subseteq[\underline{x}^0, \bar{x}^0]$.
2. $[\underline{x}^0, \bar{x}^0]\subseteq F[\underline{x}^0, \bar{x}^0]$.
3. $F_i[\underline{x}^0, \bar{x}^0]\subseteq[\underline{x}_i^0, \bar{x}_i^0]$, $i\in I$, $[\underline{x}_j^0, \bar{x}_j^0]\subseteq F_j[\underline{x}_j^0, \bar{x}_j^0]$, $j\in J$, $I\cap J=\varnothing$, $I\cup J=N$.
4. $F[\underline{x}^0, \bar{x}^0]\cap[\underline{x}^0, \bar{x}^0]=\varnothing$.
5. $f(A\bar{x}^0, B\underline{x}^0)\geq\underline{x}^0$, $\bar{x}^0\geq f(A\underline{x}^0, B\bar{x}^0)$.

The other initial conditions will be discussed in the sequel of this paper.

Algorithm 1.

$$[\underline{x}^{k+1}, \bar{x}^{k+1}]=F[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k],$$
$$F[\underline{x}^k, \bar{x}^k]=[f(A\underline{x}^k, B\bar{x}^k), f(A\bar{x}^k, B\underline{x}^k)].$$

Theorem 1. *Suppose that $f(Ax, By)$ is continuous in $[\underline{x}^0, \bar{x}^0]$ and there is $P\in R^{n\times n}$, $P\geq 0$, $\rho(P)<1$, such that*

$$f(A\bar{x}, B\underline{x})-f(A\underline{x}, B\bar{x})\leq P(\bar{x}-\underline{x}) \tag{2.1}$$

for any $[\underline{x}, \bar{x}]\subseteq[\underline{x}^0, \bar{x}^0]$. Then, when

$$F[\underline{x}^0, \bar{x}^0]\subseteq[\underline{x}^0, \bar{x}^0] \tag{2.2}$$

we have

(1) *There exists a solution x^* of (1.4) in $[\underline{x}^0, \bar{x}^0]$.*

(2) *The sequence given by Algorithm 1 satisfies*

$$[\underline{x}^{k+1}, \bar{x}^{k+1}]\subseteq[\underline{x}^k, \bar{x}^k], \tag{2.3}$$

$$x^* \in \bigcap_{k=0}^{\infty} [\underline{x}^k, \bar{x}^k], \tag{2.4}$$

$$\bar{x}^{k+1} - \underline{x}^{k+1} \leqslant P(\bar{x}^k - \underline{x}^k). \tag{2.5}$$

$[\underline{x}^k, \bar{x}^k]$ converges to x^*, which is unique in $[\underline{x}^0, \bar{x}^0]$.

(3) *The real iterative sequence*

$$x^{k+1} = f(Ax^k, Bx^k)$$

converges to x^* for any starting point $x^0 \in [\underline{x}^0, \bar{x}^0]$.

Proof. (1) Because $F[x, x] = f(Ax, Bx)$ and

$$f(A\underline{y}, B\bar{y}) \leqslant f(A\underline{x}, B\bar{x}) \leqslant f(A\bar{x}, B\underline{x}) \leqslant f(A\bar{y}, B\underline{y})$$

for arbitrary $[\underline{x}, \bar{x}] \subseteq [\underline{y}, \bar{y}]$, F is an inclusion monotonic interval extension of f. Therefore we have

$$\{f(Ax, Bx) \mid x \in [\underline{x}^0, \bar{x}^0]\} \subseteq F[\underline{x}^0, \bar{x}^0] \subseteq [\underline{x}^0, \bar{x}^0].$$

By Brouwer's fixed point theorem, there exists a solution x^* of (1.4) in $[\underline{x}^0, \bar{x}^0]$.

(2) Clearly (2.3) holds. From the inclusion monotonicity of F, as $x^* \in [\underline{x}^k, \bar{x}^k]$, we have

$$x^* = f(Ax^*, Bx^*) \in F[\underline{x}^k, \bar{x}^k],$$

$$x^* \in F[\underline{x}^k, \bar{x}^k] \cap [\underline{x}^k, \bar{x}^k] = [\underline{x}^{k+1}, \bar{x}^{k+1}].$$

From $x^* \in [\underline{x}^0, \bar{x}^0]$, (2.4) holds.

For (2.5), because of (2.2),

$$[\underline{x}^1, \bar{x}^1] = F[\underline{x}^0, \bar{x}^0] \cap [\underline{x}^0, \bar{x}^0] = F[\underline{x}^0, \bar{x}^0] \subseteq [\underline{x}^0, \bar{x}^0].$$

Let

$$[\underline{x}^k, \bar{x}^k] = F[\underline{x}^{k-1}, \bar{x}^{k-1}] \subseteq [\underline{x}^{k-1}, \bar{x}^{k-1}].$$

Then

$$[\underline{x}^{k+1}, \bar{x}^{k+1}] = F[\underline{x}^k, \bar{x}^k] \cap [\underline{x}^k, \bar{x}^k] = F[\underline{x}^k, \bar{x}^k] \cap F[\underline{x}^{k-1}, \bar{x}^{k-1}] = F[\underline{x}^k, \bar{x}^k].$$

Therefore we have

$$\bar{x}^{k+1} - \underline{x}^{k+1} = f(A\bar{x}^k, B\underline{x}^k) - f(A\underline{x}^k, B\bar{x}^k) \leqslant P(\bar{x}^k - \underline{x}^k).$$

From $\rho(P) < 1$ and (2.4), we have

$$\lim_{k\to\infty} W[\underline{x}^k, \bar{x}^k] = 0,$$

$$\lim_{k\to\infty} [\underline{x}^k, \bar{x}^k] = x^*.$$

(3) If $x^k \in [\underline{x}^k, \bar{x}^k]$, then

$$x^{k+1} = f(Ax^k, Bx^k) \in F[\underline{x}^k, \bar{x}^k] = [\underline{x}^{k+1}, \bar{x}^{k+1}].$$

Since x^k and x^* are in $[\underline{x}^k, \bar{x}^k]$, we have $\lim\limits_{k\to\infty} x^k = x^*$.

Corollary 1. Suppose that $f(Ax)$ is continuous in $[\underline{x}^0, \bar{x}^0]$ and

$$F[\underline{x}^0, \bar{x}^0] \subseteq [\underline{x}^0, \bar{x}^0].$$

Then, there exists a solution x^* of (1.4) in $[\underline{x}^0, \bar{x}^0]$ and the sequence given by Algorithm 1 satisfies

$$[\underline{x}^{k+1}, \bar{x}^{k+1}] \subseteq [\underline{x}^k, \bar{x}^k],$$

$$\underline{x}^k \to \underline{x}^*, \quad \bar{x}^k \to \bar{x}^*.$$

$\underline{x}^*$, $\bar{x}^*$ are minimal and maximal solutions of (1.4) in $[\underline{x}^0, \bar{x}^0]$.

Algorithm 2 (Convergence accelerated by the overrelaxation method).

$$[\underline{x}^{k+1}, \bar{x}^{k+1}] = L_w[\underline{x}^k, \bar{x}^k] \cap [\underline{x}^k, \bar{x}^k],$$

$$L_w[\underline{x}^k, \bar{x}^k] = [\underline{x}^k + w(f(A\underline{x}^k, B\bar{x}^k) - \underline{x}^k), \bar{x}^k + w(f(A\bar{x}^k, B\underline{x}^k) - \bar{x}^k)], \quad w \in R, w > 1.$$

Theorem 2. *Suppose that the conditions of Theorem* 1 *hold and there exists* $0 < \beta \leqslant \min_{1 \leqslant i \leqslant n} \{b_{ii}\}$, $b_{ii} = e_i^T P e_i$, *such that*

$$f(Ax, By) - f(Ax', By') \geqslant \beta(x - x')$$

for all comparable x, y *and* x', y', $x \geqslant x'$, $y' \geqslant y$. *Then the conclusions of Theorem* 1 *hold. Let* $w = 1/(1-\beta)$; *the sequence given by Algorithm* 2 *satisfies*

$$[\underline{x}^{k+1}, \bar{x}^{k+1}] \subseteq [\underline{x}^k, \bar{x}^k],$$

$$x^* = \bigcap_{k=0}^{\infty} [\underline{x}^k, \bar{x}^k], \tag{2.6}$$

$$\bar{x}^{k+1} - \underline{x}^{k+1} \leqslant C(\bar{x}^k - \underline{x}^k), \tag{2.7}$$

where $C = I - w(I - P) \geqslant 0$, $\rho(C) = \rho(P) - \beta(1 - \rho(P))/(1-\beta) < \rho(P) < 1$.

The real iterative sequence

$$x^{k+1} = x^k + w(f(Ax^k, Bx^k) - x^k) \tag{2.8}$$

converges to x^* *for any starting point* $x^0 \in [\underline{x}^0, \bar{x}^0]$.

Proof. Let

$$l(x, y) = x + w(f(Ax, By) - x).$$

Clearly $l(x, x) = x$ and (1.4) are equivalent. For any comparable x, y and x', y', $x' \leqslant x$, $y \leqslant y'$, we have

$$l(x, y) - l(x', y') \geqslant x - x' + w\beta(x - x') - w(x - x') \geqslant 0.$$

Hence $l(x, y)$ is isotone in the first vector variable and antitone in the second one, when they are comparable in $[\underline{x}^0, \bar{x}^0]$.

Because

$$L_w[\underline{x}, \bar{x}] = [l(\underline{x}, \bar{x}), l(\bar{x}, \underline{x})],$$

$$L_w[x, x] = l(x, x),$$

$$l(\underline{y}, \bar{y}) \leqslant l(\underline{x}, \bar{x}) \leqslant l(\bar{x}, \underline{x}) \leqslant l(\bar{y}, \underline{y})$$

for any $[\underline{x}, \bar{x}] \subseteq [\underline{y}, \bar{y}]$, L_w is an inclusion monotonic interval extension of l.

Because of (2.2),

$$\underline{x}^0 - l(\underline{x}^0, \bar{x}^0) = -w(f(A\underline{x}^0, B\bar{x}^0) - \underline{x}^0) \leqslant 0,$$

$$\bar{x}^0 - l(\bar{x}^0, \underline{x}^0) = -w(f(A\bar{x}^0, B\underline{x}^0) - \bar{x}^0) \geqslant 0,$$

that is

$$L_w[\underline{x}^0, \bar{x}^0] \subseteq [\underline{x}^0, \bar{x}^0].$$

According to the proof of Theorem 1, (2.6) and (2.7) hold, and the sequence given by (2.8) satisfies $x^k \in [\underline{x}^k, \bar{x}^k]$.

From Lemma 1(4) and $0 < \beta \leqslant \min_{1 \leqslant i \leqslant n} \{b_{ii}\}$ we have

$$1 - b_{ii} > 0, \quad 0 < (1 - b_{ii})/(1-\beta) \leqslant 1, \quad i = 1, 2, \cdots, n,$$

$$C = I - (I - P)/(1-\beta) \geqslant 0.$$

From Lemma 1(1) we have

$$\rho(C)=\max_{\lambda_i>0}((-\beta)/(1-\beta)+\lambda_i/(1-\beta))\geq 0,$$

where λ_i is a real eigenvalue of P. Hence

$$\rho(C)=(-\beta)/(1-\beta)+\rho(P)/(1-\beta).$$

Because of Lemma 1(3), $\beta\leq\rho(P)$,

$$0\leq\rho(C)=1-(1-\rho(P))/(1-\beta)=\rho(P)-\beta(1-\rho(P))/(1-\beta)<\rho(P)<1.$$

Algorithm 3.

$$[\underline{x}^{k+1}, \bar{x}^{k+1}]=D_w[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k],$$

$$D_w[\underline{x}^k, \bar{x}^k]=[\underline{x}^k+w(f(A\bar{x}^k, B\underline{x}^k)-\underline{x}^k), \bar{x}^k+w(f(A\underline{x}^k, B\bar{x}^k)-\bar{x}^k)],$$

where $w\in R$, $w>0$.

When $r=0$, that is, $f(Bx)$ is an antitone mapping, we have

Theorem 3. *Suppose that $f(Bx)$ is continuous in $[\underline{x}^0, \bar{x}^0]$ and there are α, $\beta>0$, such that*

$$\beta(\bar{x}-\underline{x})\geq f(B\underline{x})-f(B\bar{x})\geq\alpha(\bar{x}-\underline{x}) \tag{2.9}$$

for any $[\underline{x}, \bar{x}]\subseteq[\underline{x}^0, \bar{x}^0]$. Then, when

$$f(B\bar{x}^0)\leq\bar{x}^0, \quad \underline{x}^0\leq f(B\underline{x}^0) \tag{2.10}$$

we have

(1) *There exists a solution x^* of (1.4) in $[\underline{x}^0, \bar{x}^0]$.*

(2) *For any $w\in(0, 1/(1+\beta)]$, the sequence given by Algorithm 3 satisfies*

$$[\underline{x}^{k+1}, \bar{x}^{k+1}]\subseteq[\underline{x}^k, \bar{x}^k],$$

$$x^*\in\bigcap_{k=0}^{\infty}[\underline{x}^k, \bar{x}^k],$$

$$\bar{x}^{k+1}-\underline{x}^{k+1}\leq q(\bar{x}^k-\underline{x}^k), \quad 0<q=1-w(1+\alpha)<1. \tag{2.11}$$

The real iterative sequence

$$x^{k+1}=x^k+w(f(Bx^k)-x^k)$$

converges to x^* for any starting point $x^0\in[\underline{x}^0, \bar{x}^0]$.

Proof. Let

$$d(x)=x+w(f(Bx)-x).$$

Clearly $d(x)=x$ and (1.4) are equivalent, and

$$\begin{aligned}d(\bar{x})-d(\underline{x})&=(1-w)(\bar{x}-\underline{x})-w(f(B\underline{x})-f(B\bar{x}))\\&\geq(1-w(1+\beta))(\bar{x}-\underline{x})\geq 0\end{aligned}$$

for any $\underline{x}\leq\bar{x}$. Hence d is isotone, and $D_w[\underline{x}, \bar{x}]=[d(\underline{x}), d(\bar{x})]$ is an inclusion monotonic interval extension of d. From (2.10) we have

$$d(\underline{x}^0)-\underline{x}^0=w(f(B\underline{x}^0)-\underline{x}^0)\geq 0,$$
$$d(\bar{x}^0)-\bar{x}^0=w(f(B\bar{x}^0)-\bar{x}^0)\leq 0,$$

i.e.

$$D_w[\underline{x}^0, \bar{x}^0]\subseteq[\underline{x}^0, \bar{x}^0].$$

From (2.9) we have

$$wD_w[\underline{x}^k, \bar{x}^k]=(1-w)(\bar{x}^k-\underline{x}^k)-w(f(B\underline{x}^k)-f(B\bar{x}^k))\leq(1-w(1+\alpha))(\bar{x}^k-\underline{x}^k).$$

Following the proof of Theorem 1, we can easily complete the proof of this theorem.

Algorithm 4.

$$[\underline{x}^{k+1}, \bar{x}^{k+1}] = R_w[\underline{x}^k, \bar{x}^k] \cap [\underline{x}^k, \bar{x}^k],$$

$$R_w[\underline{x}^k, \bar{x}^k] = [\underline{x}^k + wQ(f(A\bar{x}^k, B\underline{x}^k) - \bar{x}^k), \bar{x}^k + wQ(f(A\underline{x}^k, B\bar{x}^k) - \underline{x}^k)],$$

$$w \in R,\ 0<w<1,\ Q \in R^{n\times n},\ Q \geqslant 0.$$

Theorem 4. *Suppose that $f(Ax, By)$ is continuous in $[\underline{x}^0, \bar{x}^0]$ and for any comparable x, y and x', y'*

$$f(Ax, By) - f(Ax', By') \leqslant P(y'-y) + (x-x'), \quad \forall y' \geqslant y,\ x \geqslant x',$$

$$Q(f(Ax, Bx') - f(Ax', Bx)) \geqslant \alpha(x-x'), \quad \forall x \geqslant x',$$

$$\rho(Q) < \alpha,$$

where P, $Q \in R^{n\times n}$, Q is a nonnegative, nonsingular, left subinverse of P. When

$$[\underline{x}^0, \bar{x}^0] \subseteq F[\underline{x}^0, \bar{x}^0] \tag{2.12}$$

we have: (1) *There exists a unique solution x^* of* (1.4) *in* $[\underline{x}^0, \bar{x}^0]$.

(2) *Let*

$$0 < w \leqslant \min_{1\leqslant i\leqslant n} \{1, 1/(\alpha - q_{ii})\}.$$

Then the sequence given by Algorithm 4 *satisfies*

$$[\underline{x}^{k+1}, \bar{x}^{k+1}] \subseteq [\underline{x}^k, \bar{x}^k],$$

$$x^* \in \bigcap_{k=0}^{\infty} [\underline{x}^k, \bar{x}^k];$$

$$\bar{x}^{k+1} - \underline{x}^{k+1} \leqslant G(\bar{x}^k - \underline{x}^k),$$

where $G = I - w(\alpha I - Q) \geqslant 0$, $\rho(G) < 1$.

(3) *The real iterative sequence*

$$x^{k+1} = x^k + wQ(f(Ax^k, Bx^k) - x^k)$$

converges to x^* for any starting point $x^0 \in [\underline{x}^0, \bar{x}^0]$.

Proof. Let

$$r(x, y) = x + wQ(f(Ay, Bx) - y).$$

Clearly $r(x, x) = x$ and (1.4) are equivalent, and for any comparable x, y and x', y', $x \geqslant x'$, $y \leqslant y'$,

$$r(x, y) - r(x', y') = x - x' + wQ(y'-y) - wQ(f(Ay', Bx') - f(Ay, Bx))$$

$$\geqslant x - x' + wQ(y'-y) - wQ(P(x-x') + (y'-y)) \geqslant 0.$$

Hence $r(x, y)$ is isotone in the first vector variable and antitone in the second one, when they are comparable.

By the proof of Theorem 1, we only have to prove the following

(a) R_w is an inclusion monotonic interval extension of r.

(b) $R_w[\underline{x}^0, \bar{x}^0] \subseteq [\underline{x}^0, \bar{x}^0]$.

(c) $WR_w[\underline{x}^0, \bar{x}^0] \leqslant G(\bar{x} - \underline{x})$, $\rho(G) < 1$.

For any $[\underline{x}, \bar{x}] \subseteq [\underline{y}, \bar{y}]$,

$$r(\underline{y}, \bar{y}) \leqslant r(\underline{x}, \bar{x}) \leqslant r(\bar{x}, \underline{x}) \leqslant r(\bar{y}, \underline{y}),$$

$$R_w[\underline{x}, \bar{x}] = [r(\underline{x}, \bar{x}), r(\bar{x}, \underline{x})],$$

$$R_w[x, x] = r(x, x)$$

hold, i.e. (a) holds.

Because of (2.12),

$$\bar{x}^0+wQ(f(A\underline{x}^0,\ B\bar{x}^0)-\underline{x}^0)-\bar{x}^0\leqslant 0,$$
$$\underline{x}^0+wQ(f(A\bar{x}^0,\ B\underline{x}^0)-\bar{x}^0]-\underline{x}^0\geqslant 0,$$

i.e.

$$R_w[\underline{x}^0,\ \bar{x}^0]\subseteq[\underline{x}^0,\ \bar{x}^0].$$

For (c), we have

$$WR_w[\underline{x},\ \bar{x}]=(I+wQ)(\bar{x}-\underline{x})-wQ(f(A\bar{x},\ B\underline{x})-f(A\underline{x},\ B\bar{x}))$$
$$\leqslant(I-w(\alpha I-Q))(\bar{x}-\underline{x}).$$

From Lemma 2, $G=I-w(\alpha I-Q)\geqslant 0$ and $\rho(G)<1$ hold.

According to Lemma 1 and Lemma 3, we have

Remark 1. For $\rho(Q)<\alpha$, we only have to decide whether or not there is a solution for the linear system

$$Qx<\alpha x,\quad x>0.$$

For theoretical determination of an optimum relaxation factor, we have

$$\rho(I-w_1(\alpha I-Q))\geqslant\rho(I-w_2(\alpha I-Q)),\quad w_1\leqslant w_2\leqslant\min_{1<i<n}\{1,\ 1/(\alpha-q_{ii})\}.$$

If Q is an irreducible matrix, then

$$\rho(I-w_1(\alpha I-Q))>\rho(I-w_2(\alpha I-Q)),\quad w_1<w_2\leqslant\min_{1<i<n}\{1,\ 1/(\alpha-q_{ii})\}.$$

Algorithm 5.

$$[\underline{x}^{k+1},\ \bar{x}^{k+1}]=F[\underline{x}^k,\ \bar{x}^k]\cap R_w[\underline{x}^k,\ \bar{x}^k]\cap[\underline{x}^k,\ \bar{x}^k].$$

Theorem 5. *Suppose that $f(Ax,\ By)$ is continuous in $[\underline{x}^0,\ \bar{x}^0]$ and there are $0<q<1$, $0<\alpha$, $p_j>0$, such that*

$$f_j(Ax,\ By)-f_j(Ax',\ By')\leqslant p_j(y'_j-y_j)+(x_j-x'_j),$$
$$f_j(Ax,\ Bx')-f_j(Ax',\ Bx)\geqslant p_j\alpha(x_j-x'_j),\quad p_j^{-1}<\alpha,$$
$$f_i(Ax,\ Bx')-f_i(Ax',\ Bx)\leqslant q(x_i-x'_i)$$

for any comparable x, y and x', y', $x\geqslant x'$, $y\leqslant y'$, and $i\in I$, $j\in J$, $I\cup J=N$, $I\cap J=\varnothing$. When

$$F_i[\underline{x}^0,\ \bar{x}^0]\subseteq[\underline{x}_i^0,\ \bar{x}_i^0],$$
$$[\underline{x}_j^0,\ \bar{x}_j^0]\subseteq F_j[\underline{x}^0,\ \bar{x}^0]$$

we have

(1) *There exists a unique solution x^* of (1.4) in $[\underline{x}^0,\ \bar{x}^0]$.*

(2) *Let*

$$0<w\leqslant\min_{j\in J}\{1,\ 1/(\alpha-p_j^{-1})\}.$$

The sequence given by Algorithm 5 satisfies

$$[\underline{x}^{k+1},\ \bar{x}^{k+1}]\subseteq[\underline{x}^k,\ \bar{x}^k],$$
$$x^*=\bigcap_{k=0}^{\infty}[\underline{x}^k,\ \bar{x}^k],$$
$$\bar{x}^{k+1}-\underline{x}^{k+1}\leqslant\tilde{q}(\bar{x}^k-\underline{x}^k),\quad 0<\tilde{q}<1.$$

The real iterative sequence

$$x^{k+1}=t(x^k, x^k)$$

converges to x^* *for any starting point* $x^0\in[\underline{x}^0, \bar{x}^0]$, *where*

$$t(x, y)=\begin{cases} f_i(Ax, By), & i\in I, \\ r_j(x, y), & j\in J. \end{cases}$$

Let the interval function

$$T[x, x]=\begin{cases} F_i[\underline{x}, \bar{x}], & i\in I, \\ R_{w_j}[\underline{x}, \bar{x}], & j\in J, \end{cases}$$

$$\tilde{q}=\max\{1-w(\alpha-p_j^{-1}), q\}.$$

Using Theorem 1 and Theorem 4, we can complete the proof.

Remark 2. From the inclusion monotonicity of F, D, R, T, L, we have

(i) If $F[\underline{x}^0, \bar{x}^0]\cap[\underline{x}^0, \bar{x}^0]=\varnothing$, then $x^*\bar{\in}[\underline{x}^0, \bar{x}^0]$.

(ii) If $D_w[\underline{x}^0, \bar{x}^0]\cap[\underline{x}^0, \bar{x}^0]=\varnothing$, then $x^*\bar{\in}[\underline{x}^0, \bar{x}^0]$.

(iii) If $R_w[\underline{x}^0, \bar{x}^0]\cap[\underline{x}^0, \bar{x}^0]=\varnothing$, then $x^*\bar{\in}[\underline{x}^0, \bar{x}^0]$.

(iv) If $F[\underline{x}^0, \bar{x}^0]\cap R_w[\underline{x}^0, \bar{x}^0]\cap[\underline{x}^0, \bar{x}^0]=\varnothing$, then $x^*\bar{\in}[\underline{x}^0, \bar{x}^0]$.

$$L_w[\underline{x}, \bar{x}]\subseteq F[\underline{x}, \bar{x}], \text{ as } w\geqslant 1.\ F[\underline{x}, \bar{x}]\subseteq[\underline{x}, \bar{x}].$$

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