THE CHEBYSHEV SPECTRAL METHOD FOR BURGERS-LIKE EQUATIONS*

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Abstract

The Chebyshev polynomials have good approximation properties which are not affected by boundary values. They have higher resolution near the boundary than in the interior and are suitable for problems in which the solution changes rapidly near the boundary. Also, they can be calculated by FFT. Thus they are used mostly for initial-boundary value problems for P. D. E. 's(see [1, 3—4, 6, 8—11]). Maday and Quarteroni[8] discussed the convergence of Legendre and Chebyshev spectral approximations to the steady Burgers equation. In this paper we consider Burgers-like equations

$$\begin{cases} \partial_t u+F(u)_x-\nu u_{xx}=0, & -1\leqslant x\leqslant 1,\ 0<t\leqslant T,\\ u(-1,t)=u(1,t)=0, & 0\leqslant t\leqslant T,\\ u(x,0)=u_0(x), & -1\leqslant x\leqslant 1,\end{cases} \tag{0.1}$$

where $F\in C(\mathbf{R})$ and there exists a positive function $A\in C(\mathbf{R})$ and a constant $p>1$ such that

$$|F(z+y)-F(z)|\leqslant A(z)(|y|+|y|^p).$$

We develop a Chebyshev spectral scheme and a pseudospectral scheme for solving (0.1) and establish their generalized stability and convergence.

§ 1. Notations and Lemmas

Let $I=(-1,1)$, and let $L^2(I)$ be equipped with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Suppose $\omega(x)=(1-x^2)^{-1/2}$; set

$$L^2_\omega(I)=\{v: I\to R\,|\,v \text{ is measurable and } (v,v)_\omega<\infty\},$$

where
$$(u,v)_\omega=\int_I u(x)v(x)\omega(x)dx,\quad \|u\|_\omega=(u,u)_\omega^{1/2}.$$

For any positive integer m, define

$$\|u\|^2_{m,\omega}=\sum_{j=0}^{m}\left\|\frac{d^ju}{dx^j}\right\|^2_\omega,\ H^m_\omega(I)=\{v\in L^2_\omega(I)\,|\,\|v\|_{m,\omega}<\infty\},$$

$$H^1_{0,\omega}(I)=\{v\in H^1_\omega(I)\,|\,v(-1)=v(1)=0\}.$$

For any positive integer N, let S_N be the space of algebraic polynomials of degree at most N. Set

$$V_N=\{\varphi\in S_N\,|\,\varphi(-1)=\varphi(1)=0\}=S_N\cap H^1_{0,\omega}(I).$$

Let $P_N: L^2_\omega(I)\to V_N$ be the L^2_ω-orthogonal projection operator, i.e.,

$$(P_Nv,\varphi)_\omega=(v,\varphi)_\omega,\quad \forall\varphi\in V_N,$$

and $P_{1,N}: H^1_{0,\omega}(I)\to V_N$ be as follows:

$$((P_{1,N}v-v)_x,(\varphi\omega)_x)=0,\quad \forall\varphi\in V_N.$$

Denote by $\{x_j,\omega_j\}$ the nodes and weights of the Gauss-Lobatto integration

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formula, where $x_j=\cos\frac{\pi}{N}j\,(0\leqslant j\leqslant N)$, $\omega_0=\omega_N=\frac{\pi}{2N}$ and $\omega_j=\frac{\pi}{N}$, $1\leqslant j\leqslant N-1$. Then

$$\int_I f(x)\,\omega(x)\,dx=\sum_{j=0}^{N} f(x_j)\,\omega_j,\quad \forall f\in S_{2N-1}. \tag{1.1}$$

Let P_c: $C(\bar{I})\to S_N$ be the interpolation operator: $P_c u(x_j)=u(x_j)$ $(0\leqslant j\leqslant N)$. Introduce the discrete inner product and norm

$$(u,\ v)_{N,\omega}=\sum_{j=0}^{N} u(x_j)\,v(x_j)\,\omega_j,\quad \|u\|_{N,\omega}=(u,u)_{N,\omega}^{1/2}.$$

The constants c in the following lemmas are independent of N and of the function v, which may be different in different cases.

Lemma 1. *If* $v\in H^1_{0,\omega}(I)$, *then*

$$\|v\omega^2\|_\omega\leqslant|v|_{1,\omega}. \tag{1.2}$$

Proof. Let $g(t)=t^{-1}\int_0^t f(s)\,ds$. We get from Theorem 4.1 in Chapter 3 of [7]

$$\int_0^\infty |g(t)|^2 t^{-1/2}dt\leqslant\frac{16}{9}\int_0^\infty |f(t)|^2 t^{-1/2}dt.$$

Set $t=1+x$; then

$$\int_{-1}^{0}|g(1+x)|^2(1+x)^{-1/2}dx\leqslant\frac{16}{9}\int_{-1}^{\infty}|f(1+x)|^2(1+x)^{-1/2}dx.$$

Now take

$$f(1+x)=\begin{cases} v_x(x), & -1\leqslant x\leqslant 0,\\ 0, & x>0.\end{cases}$$

Then $g(1+x)=(1+x)^{-1}\int_{-1}^{x}f(1+\xi)\,d\xi=(1+x)^{-1}v(x)$ and so

$$\int_{-1}^{0}|v(x)|^2\omega^5(x)\,dx\leqslant\frac{16}{9}\sqrt{2}\int_{-1}^{0}|v_x(x)|^2\omega(x)\,dx.$$

Similarly we get the result on the interval $[0,\ 1]$. Therefore

$$\|v\omega^2\|^2_\omega\leqslant\frac{16}{9}\sqrt{2}\ |v|^2_{1,\omega},$$

which leads to $\lim\limits_{x\to\pm1}v^2(x)\,\omega^3(x)=0$, and we get from integration by parts

$$\int_I v_x(v\omega)_x\,dx=\int_I |v_x|^2\omega\,dx-\frac{1}{2}\int_I (1+2x^2)\,v^2\omega^5dx, \tag{1.3}$$

$$\int_I v_x(v\omega)_x dx=\int_I |(v\omega)_x|^2\omega^{-1}dx+\frac{1}{2}\int_I v^2\omega^5dx, \tag{1.4}$$

Subtracting (1.4) from (1.3) yields

$$\int_I |v_x|^2\omega\,dx-\int_I (1+x^2)\,v^2\omega^5\,dx=\int_I |(v\omega)_x|^2\omega^{-1}dx\geqslant 0,$$

and so (1.2) follows.

Lemma 2. *If* $v\in H^1_{0,\omega}(I)$, *then*

$$\int_I v_x(v\omega)_x dx\geqslant\frac{1}{4}\ \|v\|^2_{1,\omega}. \tag{1.5}$$

Proof. We get from (1.3) and (1.4)

$$\int_I v_x(v\omega)_x dx=\frac{1}{4}\int_I |v_x|^2\omega\,dx+\frac{1}{4}\int_I v^2\omega^3\,dx+\frac{3}{4}\int_I |(v\omega)_x|^2\omega^{-1}dx\geqslant\frac{1}{4}\ \|v\|^2_{1,\omega}.$$

Lemma 3. *If $u\in L^2_\omega(I)$ and $v\in H^1_{0,\omega}(I)$, then*

$$|(u,\ (v\omega)_x)|\leqslant 2\|u\|_\omega|v|_{1,\omega}. \tag{1.6}$$

Proof. We get from Lemma 1

$$\begin{aligned}|(u,\ (v\omega)_x)|&\leqslant|(u,\ v_x)_\omega|+|(u,\ xv\omega^2)_\omega|\\&\leqslant\|u\|_\omega|v|_{1,\omega}+\|u\|_\omega\|v\omega^2\|_\omega\leqslant 2\|u\|_\omega|v|_{1,\omega}.\end{aligned}$$

Lemma 4[8,9]. If $v\in H^m_\omega(I)$, then

$$\|P_Nv-v\|_\omega\leqslant cN^{-m}\|v\|_{m,\omega}. \tag{1.7}$$

If $0\leqslant j\leqslant m, m>1/2$ and $v\in H^m_\omega(I)$, then

$$\|P_cv-v\|_{j,\omega}\leqslant cN^{2j-m}\|v\|_{m,\omega}. \tag{1.8}$$

If $m\geqslant 1$ and $v\in H^1_{0,\omega}(I)\cap H^m_\omega(I)$, then

$$\|P_{1,N}v-v\|_{j,\omega}\leqslant cN^{j-m}\|v\|_{m,\omega},\quad j=0,\ 1. \tag{1.9}$$

Lemma 5. *If $v\in S_N$, then*

$$\|v\|_{L^\infty}\leqslant N^{1/2}\|v\|_\omega. \tag{1.10}$$

Proof. Let $T_k(x)$ be the Chebyshev polynomial of degree k and

$$v(x)=\sum_{k=0}^N\sqrt{\frac{2}{\pi c_k}}\,a_kT_k(x),$$

where $c_0=2,\ c_k=1\ (k\geqslant 1)$. Because $|T_k(x)|\leqslant 1$, we have

$$\|v\|^2_{L^\infty}\leqslant\frac{2}{\pi}\left\{\sum_{k=0}^N\frac{1}{c_k}\right\}\left\{\sum_{k=0}^N a_k^2\right\}\leqslant N\|v\|^2_\omega.$$

Lemma 6. *If $u,\ v\in S_N$, then*

$$\|v\|_\omega\leqslant\|v\|_{N,\omega}\leqslant\sqrt{2}\,\|v\|_\omega, \tag{1.11}$$

$$|(u,\ v)_{N,\omega}-(u,\ v)_\omega|\leqslant cN^{-m}\|u\|_{m,\omega}\|v\|_\omega. \tag{1.12}$$

Proof. Let $\varphi_k(x)=\sqrt{\dfrac{2}{\pi c_k}}\,T_k(x),\ c_0=2,\ c_k=1\ (k\geqslant 1),\ u=\sum\limits_{k=0}^N a_k\varphi_k(x)$ and $v=\sum\limits_{k=0}^N b_k\varphi_k(x)$. We get from (1.1)

$$\|v\|^2_{N,\omega}=\left\|\sum_{k=0}^{N-1}a_k\varphi_k\right\|^2_\omega+|a_N|^2\|\varphi_N\|^2_{N,\omega}=\sum_{k=0}^{N-1}|a_k|^2+2|a_N|^2=\|v\|^2_\omega+|a_N|^2.$$

Thus (1.11) holds. Also we have

$$\begin{aligned}|a_N|&=\left|\sqrt{\frac{2}{\pi}}\int_{-1}^1u(x)T_N(x)\omega(x)dx\right|=\left|\sqrt{\frac{2}{\pi}}\int_0^\pi u(\cos\theta)\cos N\theta\,d\theta\right|\\&=\left|\frac{1}{\sqrt{2\pi}}\int_{-\pi}^\pi u(\cos\theta)e^{-iN\theta}\,d\theta\right|=\left|\frac{1}{\sqrt{2\pi}N^m}\int_{-\pi}^\pi\partial_\theta^m u(\cos\theta)e^{-iN\theta}d\theta\right|\leqslant cN^{-m}\|u\|_{m,\omega}.\end{aligned}$$

Therefore

$$|(u,\ v)_{N,\omega}-(u,\ v)_\omega|=|a_Nb_N|\leqslant cN^{-m}\|u\|_{m,\omega}\|v\|_\omega.$$

Let τ be the mesh size in variable t, and $u^k(x)=u(x,\ k\tau)$, which is denoted by u^k for simplicity. We define $u^k_t=\dfrac{1}{\tau}(u^{k+1}-u^k),\ u^k_{\bar t}=\dfrac{1}{\tau}(u^k-u^{k-1}),\ u^k_{\hat t}=\dfrac{1}{2\tau}(u^{k+1}-u^{k-1})$ and $u^{\hat k}=\dfrac{1}{2}(u^{k+1}+u^{k-1})$ and use the notations

$$\|u\|_{L^\infty}=\max_{k\tau\leqslant T}\|u^k\|_{L^\infty},\quad \|u\|_{m,\omega}=\max_{k\tau\leqslant T}\|u^k\|_{m,\omega}.$$

Lemma 7[5]. *If the following conditions hold*

(i) ρ, M_1, M_2 *and* p *are positive constants,* $p>1$ *and* E^k *is a nonnegative function,*

(ii) $E^n\leqslant\rho+M_1\tau\sum_{k=0}^{n-1}(E^k+M_2(E^k)^p)$,

(iii) $E^0\leqslant\rho,\ \rho e^{2M_1T}\leqslant M_2^{\frac{1}{1-p}}$.

Then for all $n\tau\leqslant T$,

$$E^n\leqslant\rho e^{2M_1n\tau}.$$

§ 2. The Stability and Convergence of the Spectral Scheme

The spectral scheme for problem (0.1) is to find u_N^k in V_N such that

$$\begin{cases}u_{Nt}^k+P_NF(u_N^k)_x-\nu P_Nu_{Nxx}^{\hat{k}}=0, & k\geqslant1,\\ u_N^1=P_N(u_0+\tau\partial_tu(0)),\\ u_N^0=P_Nu_0.\end{cases}\tag{2.1}$$

Now we consider the generalized stability of (2.1). Suppose that u_N^k and the right term in (2.1) have respectively the error $\tilde{u}^k$ and $\tilde{f}^k\in V_N$. Then

$$(\tilde{u}_t^k,\varphi)_\omega-(\tilde{F}^k,(\varphi\omega)_x)+\nu(\tilde{u}_x^{\hat{k}},(\varphi\omega)_x)=(\tilde{f}^k,\varphi)_\omega,\quad\forall\varphi\in V_N,\tag{2.2}$$

where $\tilde{F}^k=F(u_N^k+\tilde{u}^k)-F(u_N^k)$. We take $\varphi=\tilde{u}^{\hat{k}}$ in (2.2) and get from (1.5)

$$\frac{1}{2}(\|\tilde{u}^k\|_\omega^2)_{\hat{t}}+\frac{\nu}{4}\|\tilde{u}^{\hat{k}}\|_{1,\omega}^2\leqslant(\tilde{f}^k,\tilde{u}^{\hat{k}})_\omega+(\tilde{F}^k,(\tilde{u}^{\hat{k}}\omega)_x).\tag{2.3}$$

Summing the formula (2.3) for all $1\leqslant k\leqslant n-1$, we have

$$\|\tilde{u}^n\|_\omega^2+\|\tilde{u}^{n-1}\|_\omega^2+\nu\tau\sum_{k=1}^{n-1}\|\tilde{u}^{\hat{k}}\|_{1,\omega}^2\leqslant\|\tilde{u}^0\|_\omega^2+\|\tilde{u}^1\|_\omega^2+4\tau\sum_{k=1}^{n-1}\{(\tilde{f}^k,\tilde{u}^{\hat{k}})_\omega+(\tilde{F}^k,(\tilde{u}^{\hat{k}}\omega)_x)\}.\tag{2.4}$$

We get from (1.6) and (1.10)

$$\begin{aligned}|4(\tilde{F}^k,(\tilde{u}^{\hat{k}}\omega)_x)|&\leqslant8\|\tilde{F}^k\|_\omega|\tilde{u}^{\hat{k}}|_{1,\omega}\leqslant\frac{\nu}{2}\|\tilde{u}^{\hat{k}}\|_{1,\omega}^2+\frac{32}{\nu}\|\tilde{F}^k\|_\omega^2\\&\leqslant\frac{\nu}{2}\|\tilde{u}^{\hat{k}}\|_{1,\omega}^2+c(A(u_N^k),\nu)(\|\tilde{u}^k\|_\omega^2+N^{p-1}\|\tilde{u}^k\|_\omega^{2p}),\end{aligned}\tag{2.5}$$

and also we have

$$|4(\tilde{f}^k,\tilde{u}^{\hat{k}})_\omega|\leqslant2\|\tilde{f}^k\|_\omega^2+\|\tilde{u}^{k+1}\|_\omega^2+\|\tilde{u}^{k-1}\|_\omega^2.\tag{2.6}$$

Provided $\tau<1/2$, it comes from (2.4), (2.5) and (2.6) that

$$\begin{aligned}\|\tilde{u}^n\|_\omega^2+\frac{\nu}{2}\tau\sum_{k=1}^{n-1}\|\tilde{u}^{\hat{k}}\|_{1,\omega}^2&\leqslant2\Big(\|\tilde{u}^0\|_\omega^2+\|\tilde{u}^1\|_\omega^2+2\tau\sum_{k=1}^{n-1}\|\tilde{f}^k\|_\omega^2\Big)\\&\quad+\tau\sum_{k=1}^{n-1}c(\|u_N\|_{L^\infty},\nu)(\|\tilde{u}^k\|_\omega^2+N^{p-1}\|\tilde{u}^k\|_\omega^{2p}).\end{aligned}\tag{2.7}$$

Let

$$\rho^n=2\Big(\|\tilde{u}^0\|_\omega^2+\|\tilde{u}^1\|_\omega^2+2\tau\sum_{k=1}^{n-1}\|\tilde{f}^k\|_\omega^2\Big),$$

$$E^n=\|\tilde{u}^n\|_\omega^2+\frac{\nu}{2}\tau\sum_{k=1}^{n-1}\|\tilde{u}^{\hat{k}}\|_{1,\omega}^2.$$

Then (2.7) leads to

$$E^n\leqslant\rho^n+\tau\sum_{k=1}^{n-1}c(\|u_N\|_{L^\infty},\nu)(E^k+N^{p-1}(E^k)^p).$$

Finally we get the following result by Lemma 7.

Theorem 1. *If τ is suitably small, then there exist positive constants c and δ depending on $\|u_N\|_{L^\infty}$ and ν such that when $\rho^{[T/\tau]}\leqslant\delta N^{-1}$, we have for all $n\tau\leqslant T$*

$$E^n\leqslant \rho^n e^{cn\tau}.$$

We next consider the convergence of (2.1).

Theorem 2. *If $u\in C^1(0, T; H^1_{0,\omega}(I)\cap H^m_\omega(I))(m\geqslant1)$ and $\partial_t^3u\in H^1(0, T;L^2_\omega(I))$, then there exist positive constants c and δ depending on u such that when τ is suitably small and $\tau^2+N^{-m}\leqslant\delta N^{-1/2}$, we have for all $n\tau\leqslant T$*

$$\|u_N^n-u^n\|_\omega\leqslant c\{\tau^2+N^{-m}\}.$$

Proof. Setting $w^k=P_{1,N}u^k$ and $e^k=u_N^k-w^k$, we get from (0.1) and (2.1)

$$(e_{\hat t}^k, \varphi)_\omega-(F(w^k+e^k)-F(w^k), (\varphi\omega)_x)+\nu(e_x^{\hat k}, (\varphi\omega)_x)$$
$$=(\partial_t u^{\hat k}-w_{\hat t}^k, \varphi)_\omega+\Big(F(w^k)-F(u^k)-\frac{\tau^2}{2}F(u^k)_{tt}, (\varphi\omega)_x\Big),\quad \forall\varphi\in V_N. \tag{2.8}$$

By Taylor's formula we have

$$\tau\sum_{k=1}^{n-1}\|\partial_t u^{\hat k}-w_{\hat t}^k\|_\omega^2\leqslant2\tau\sum_{k=1}^{n-1}(\|\partial_t u^{\hat k}-u_{\hat t}^k\|_\omega^2+\|u_{\hat t}^k-P_{1,N}u_{\hat t}^k\|_\omega^2)$$
$$\leqslant c\tau^4\|\partial_t^3u\|^2_{L^2(0,T;L^2_\omega)}+cN^{-2m}\|\partial_t u\|^2_{L^2(0,T;H^m_\omega)},$$
$$\|F(u^k)_{tt}\|_\omega\leqslant c(\|u\|_{W^{1,\infty}(0,T;L^\infty)})(\|\partial_t^2u\|_{L^\infty(0,T;L^2_\omega)}+1).$$

Moreover

$$\|F(w^k)-F(u^k)\|_\omega\leqslant c(\|u\|_{L^\infty(0,T;H^1_\omega)})\|P_{1,N}u^k-u^k\|_\omega\leqslant cN^{-m}\|u\|_{m,\omega},$$
$$\|e^0\|_\omega\leqslant\|P_Nu_0-u_0\|_\omega+\|P_{1,N}u_0-u_0\|_\omega\leqslant cN^{-m}\|u_0\|_{m,\omega},$$
$$\|e^1\|_\omega\leqslant\|e^0\|_\omega+\tau\|(P_N-P_{1,N})\partial_t u(0)\|_\omega+\frac{1}{2}\tau^2\|P_{1,N}\partial_t^2u\|_{L^\infty(0,T;L^2_\omega)}$$
$$\leqslant c(\|u_0\|_{m,\omega}, \|\partial_t u(0)\|_{m,\omega}, \|\partial_t^2u\|_{L^\infty(0,T;L^2_\omega)})\{\tau^2+N^{-m}\}.$$

Finally we apply Theorem 1 to (2.8) and use the triangle inequality and (1.9) to complete the proof.

§ 3. The Stability and Convergence of the Pseudospectral Scheme

The pseudospectral scheme for problem (0.1) is to find u_c^k in V_N such that

$$\begin{cases} u_{c\hat t}^k(x_j)+(P_cF(u_c^k))_x(x_j)-\nu u_{cxx}^{\hat k}(x_j)=0, & 1\leqslant j\leqslant N-1,\ k\geqslant1,\\ u_c^1(x_j)=u_0(x_j)+\tau\partial_t u(x_j, 0), & 0\leqslant j\leqslant N,\\ u_c^0(x_j)=u_0(x_j), & 0\leqslant j\leqslant N.\end{cases} \tag{3.1}$$

Now we consider the generalized stability of (3.1). Suppose that u_c^k and the right term in (3.1) have respectively the error $\tilde u^k$ and $\tilde f^k\in V_N$. Then

$$(\tilde u_{\hat t}^k, \varphi)_{N,\omega}-(P_c\tilde F^k, (\varphi\omega)_x)+\nu(\tilde u_x^{\hat k}, (\varphi\omega)_x)=(\tilde f^k, \varphi)_{N,\omega},\quad \forall\varphi\in V_N, \tag{3.2}$$

where $\tilde F^k=F(u_c^k+\tilde u^k)-F(u_c^k)$. We have from Lemma 6

$$\|P_c\tilde F^k\|_\omega^2\leqslant\|\tilde F^k\|_{N,\omega}^2\leqslant c(A(u_c^k), \nu)(\|\tilde u^k\|_\omega^2+N^{p-1}\|\tilde u^k\|_\omega^{2p}).$$

Let
$$\rho^n=4\Big(\|\tilde u^0\|_\omega^2+\|\tilde u^1\|_\omega^2+\tau\sum_{k=1}^{n-1}\|\tilde f^k\|_{N,\omega}^2\Big),$$
$$E^n=\|\tilde u^n\|_\omega^2+\frac{\nu}{2}\tau\sum_{k=1}^{n-1}\|\tilde u^{\hat k}\|_{1,\omega}^2.$$

By an argument similar to that of Theorem 1 we get the following result.

Theorem 3. *If τ is suitably small, then there exist positive constants c and δ depending on $\|u_0\|_{L^\infty}$ and ν such that when $\rho^{[T/\tau]}\leqslant\delta N^{-1}$, we have for all $n\tau\leqslant T$*

$$E^n\leqslant\rho^n e^{cn\tau}$$

Theorem 4. *If the conditions of Theorem 2 are fulfilled, then there exist positive constants c and δ depending on u such that when τ is suitably small and $\tau^2+N^{-m}\leqslant\delta N^{-1/2}$, we have for all $n\tau\leqslant T$*

$$\|u_c^n-u^n\|_\omega\leqslant c\{\tau^2+N^{-m}\}.$$

Proof. Setteng $w^k=P_{1,N}u^k$ and $e^k=u_c^k-w^k$, we get from (0.1) and (3.1)

$$(e_{\hat{t}}^k,\varphi)_{N,\omega}-(P_c(F(w^k+e^k)-F(w^k)),(\varphi\omega)_x)+\nu(e_x^{\hat{k}},(\varphi\omega)_x)$$
$$=(\partial_t u^{\hat{k}},\varphi)_\omega-(w_{\hat{t}}^k,\varphi)_{N,\omega}+(P_cF(w^k)-F(u^k)-\frac{1}{2}\tau^2F(u^k)_{tt},(\varphi\omega)_x),\quad\forall\varphi\in V_N. \tag{3.3}$$

According to (1.12), we have

$$|(\partial_t u^{\hat{k}},\varphi)_\omega-(w_{\hat{t}}^k,\varphi)_{N,\omega}|\leqslant|(\partial_t u^{\hat{k}}-w_{\hat{t}}^k,\varphi)_\omega|+|(w_{\hat{t}}^k,\varphi)_\omega-(w_{\hat{t}}^k,\varphi)_{N,\omega}|$$
$$\leqslant\|\partial_t u^{\hat{k}}-w_{\hat{t}}^k\|_\omega^2+cN^{-2m}\|w_{\hat{t}}^k\|_{m,\omega}^2+\|\varphi\|_\omega^2.$$

Moreover

$$\|P_cF(w^k)-F(u^k)\|_\omega\leqslant\|F(w^k)-F(u^k)\|_{N,\omega}+\|P_cF(u^k)-F(u^k)\|_\omega$$
$$\leqslant c(\|u^k\|_{1,\omega})N^{-m}\|u^k\|_{m,\omega}+cN^{-m}\|F(u^k)\|_{m,\omega}\leqslant c(\|u^k\|_{1,\omega})N^{-m}\|u^k\|_{m,\omega}.$$

The remaining work is similar to that in the proof of Theorem 2.

Remark. If $u(-1,t)=g_{-1}(t)$ and $u(1,t)=g_1(t)$, Theorems 2 and 4 still hold.

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