THE CONVERGENCE OF NUMERICAL METHOD FOR NONLINEAR SCHRÖDINGER EQUATION*

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§1. Introduction

In the past few years interest has substantially increased in the solutions of partial differential equations governing nonlinear waves in dispersive media, while considerable literature has grown dealing with the numerical approximations of such problems. One of nonlinear wave equations is nonlinear Schrödinger equation whose solution is a complex field governing the evolution of any weakly nonlinear, strongly dispersive, almost monochromatic wave (see Zakharov (1968), Hasimoto and Ono (1972), Davey (1972) and Yuen and Lake (1980)). The pure initial value problem was exactly solved by Zakharov and Shabat (1972) using the inverse scattering method when the initial condition vanishes for sufficiently large $|x|$. For more general initial conditions the theoretical solution of the nonlinear Schrödinger equation is unknown. From the numerical point of view, Ablowitz and Ladik (1976) employed a difference scheme for the numerical solution of the nonlinear Schrödinger equation. Other methods were given by Yuen and Lake (1975), Kuo Pen-yu (1976), Yuen and Ferguson (1978), Yuen and Lake (1980) and Defour, Forten and Payne (1981). Recently Griffiths, Mitchell and Morris (1982) proposed a prediction-correction scheme which does not need nonlinear iteration. If we choose the parameter suitably in that scheme, then the scheme is stable and has high order accuracy. The numerical results showed the advantage of that method. For the strict error estimations, Kuo Pen-yu (1979) gave a proof for the semi-discrete scheme. Recently Zhu You-lan (1983) considered an implicit scheme and gave its convergence.

This paper is devoted to the convergence of some numerical methods such as the Crank-Nicolson method and the prediction-correction method, the finite difference scheme and the Galerkin method.

§2. Crank-Nicolson Method

We consider the following problem

$$\begin{cases} i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\alpha w|w|^2=0, & x\in\mathbf{R},\ 0<t<T, \\ w(x,0)=w_0(x), & x\in\mathbf{R}, \end{cases} \tag{1}$$

where $\alpha>0$. We suppose that for all $t\geqslant 0$, $w(x,t)\in L^2(\mathbf{R})$ and for all $x\in\mathbf{R}$, $0\leqslant t\leqslant T$,

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$w(x, t)$ is bounded,

$$\lim_{|x|\to\infty} w(x, t)=0, \quad \lim_{|x|\to\infty} \frac{\partial w}{\partial x}(x, t)=0.$$

Let $w(x, t)=u(x, t)+iv(x, t)$ where $u(x, t)$ and $v(x, t)$ are real value functions, then (1) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial^2 v}{\partial x^2}+\alpha v|w|^2=0, & x\in \mathbb{R},\ 0<t\leqslant T, \\ \dfrac{\partial v}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}-\alpha u|w|^2=0, & x\in \mathbb{R},\ 0<t\leqslant T, \\ u(x, 0)=u_0(x), & x\in \mathbb{R}, \\ v(x, 0)=v_0(x), & x\in \mathbb{R}. \end{cases} \tag{2}$$

Let

$$\|w(t)\|^2_{L^2}=\int_{\mathbb{R}} [|u(x, t)|^2+|v(x, t)|^2]dx,$$

then we have

$$\|w(t)\|^2_{L^2}=\|w_0\|^2_{L^2}, \tag{3}$$

and

$$\left\|\frac{\partial w(t)}{\partial x}\right\|^2_{L^2}-\frac{\alpha}{2}\|w(t)\|^4_{L^4}=\left\|\frac{\partial w_0}{\partial x}\right\|^2_{L^2}-\frac{\alpha}{2}\|w_0\|^4_{L^4}. \tag{4}$$

For the numerical solution of (2), we take h and k to be the mesh spacings for variables x and t respectively, $x_j=jh$, $j=0, \pm 1, \pm 2, \cdots$, $\mathbb{R}_h=\{x_j\}$. Let $\eta^n(x)$ be the value of the function η at $x\in \mathbb{R}_h$ and $t=nk$. We introduce the following notations

$$\eta^n_t(x)=\frac{1}{k}(\eta^{n+1}(x)-\eta^n(x)),$$

$$\eta^n_x(x)=\frac{1}{h}(\eta^n(x+h)-\eta^n(x)),$$

$$\eta^n_{\bar{x}}(x)=\eta^n_x(x-h),$$

and

$$\eta^n_{x\bar{x}}(x)=[\eta^n_x(x)]_{\bar{x}}.$$

We define the discrete scalar product and the norm as follows

$$(\eta^n, \xi^n)=h\sum_{x\in \mathbb{R}_h} \eta^n(x)\xi^n(x),$$

$$\|\eta^n\|^2=(\eta^n, \eta^n).$$

It can be proved that

$$(\eta^n_{x\bar{x}}, \xi^n)=(\xi^n_{x\bar{x}}, \eta^n), \tag{5}$$

$$(\eta^n_t, \eta^n+\eta^{n+1})=[\|\eta^n\|^2]_t, \tag{6}$$

$$2(\eta^n_t, \eta^n)=[\|\eta^n\|^2]_t-\tau\|\eta^n_t\|^2, \tag{7}$$

and

$$\|\eta^n\xi^n\|\leqslant\frac{1}{h}\|\eta^n\|\|\xi^n\|. \tag{8}$$

Let $U^n(x)$, $V^n(x)$ and $W^n(x)$ be the approximations of $u^n(x)$, $v^n(x)$ and $w^n(x)$ respectively. The Crank-Nicolson scheme for solving (2) is

$$\begin{cases} U_t^n(x)+\frac{1}{2}(V_{x\bar{x}}^n(x)+V_{x\bar{x}}^{n+1}(x))+\frac{\alpha}{8}(V^n(x)+V^{n+1}(x))\,|W^n(x) \\ \quad +W^{n+1}(x)|^2=0, \quad x\in \mathrm{R}_h,\ n\geq 0, \\ V_t^n(x)-\frac{1}{2}(U_{x\bar{x}}^n(x)+U_{x\bar{x}}^{n+1}(x))-\frac{\alpha}{8}(U^n(x)+U^{n+1}(x))\,|W^n(x) \\ \quad +W^{n+1}(x)|^2=0, \quad x\in \mathrm{R}_h,\ n\geq 0, \\ U^0(x)=u_0(x), \quad x\in \mathrm{R}_h, \\ V^0(x)=v_0(x), \quad x\in \mathrm{R}_h. \end{cases} \tag{9}$$

By taking the scalar products of the first two equations with $(U^n(x)+U^{n+1}(x))$ and $(V^n(x)+V^{n+1}(x))$ respectively and summing up the two results, from (5) and (6) we obtain

$$\|W^n\|^2=\|W^0\|^2, \quad n\geq 1, \tag{10}$$

which is a discrete analogy of (3).

Now we are going to estimate the error. Let

$$U^n(x)=u^n(x)+\tilde{u}^n(x),$$
$$V^n(x)=v^n(x)+\tilde{v}^n(x),$$

and

$$W^n(x)=w^n(x)+\tilde{w}^n(x).$$

Equality (10) implies

$$\|\tilde{w}^n\|^2\leq 2\|w_0\|^2, \quad \text{for } n\geq 0. \tag{11}$$

The errors satisfy the following equation

$$\begin{cases} \tilde{u}_t^n(x)+\frac{1}{2}(\tilde{v}_{x\bar{x}}^n(x)+\tilde{v}_{x\bar{x}}^{n+1}(x))+\frac{\alpha}{8}(\tilde{v}^n(x)+\tilde{v}^{n+1}(x))\,|w^n(x)+w^{n+1}(x) \\ \quad +\tilde{w}^n(x)+\tilde{w}^{n+1}(x)|^2+G_1^n(x)+G_2^n(x)=\tilde{f}^n(x), \quad x\in \mathrm{R},\ n\geq 0, \\ \tilde{v}_t^n(x)-\frac{1}{2}(\tilde{u}_{x\bar{x}}^n(x)+\tilde{u}_{x\bar{x}}^{n+1}(x))-\frac{\alpha}{8}(\tilde{u}^n(x)+\tilde{u}^{n+1}(x))\,|w^n(x)+w^{n+1}(x) \\ \quad +\tilde{w}^n(x)+\tilde{w}^{n+1}(x)|^2+G_3^n(x)+G_4^n(x)=\tilde{g}^n(x), \quad x\in \mathrm{R}_h,\ n\geq 0, \\ \tilde{u}^0(x)=0, \quad x\in \mathrm{R}_h, \\ \tilde{v}^0(x)=0, \quad x\in \mathrm{R}_h, \end{cases} \tag{12}$$

where $\tilde{f}^n(x)$ and $\tilde{g}^n(x)$ are truncation errors,

$$G_1^n(x)=\frac{\alpha}{4}(u^n(x)+u^{n+1}(x))\,(v^n(x)+v^{n+1}(x))\,(\tilde{u}^n(x)+\tilde{u}^{n+1}(x))$$
$$\quad +\frac{\alpha}{4}(v^n(x)+v^{n+1}(x))^2(\tilde{v}^n(x)+\tilde{v}^{n+1}(x)),$$

$$G_2^n(x)=\frac{\alpha}{8}(v^n(x)+v^{n+1}(x))\,|\tilde{w}^n(x)+\tilde{w}^{n+1}(x)|^2,$$

$$G_3^n(x)=-\frac{\alpha}{4}(v^n(x)+v^{n+1}(x))\,(u^n(x)+u^{n+1}(x))\,(\tilde{v}^n(x)+\tilde{v}^{n+1}(x))$$
$$\quad -\frac{\alpha}{4}(u^n(x)+u^{n+1}(x))^2(\tilde{u}^n(x)+\tilde{u}^{n+1}(x)),$$

$$G_4^n(x)=-\frac{\alpha}{8}(u^n(x)+u^{n+1}(x))\,|\tilde{w}^n(x)+\tilde{w}^{n+1}(x)|^2.$$

By taking the scalar products of the first two equations of (12) with $\tilde{u}^n(x)+\tilde{u}^{n+1}(x)$ and $\tilde{v}^n(x)+\tilde{v}^{n+1}(x)$ respectively and summing up the two results, from (5) and (6), we obtain

$$
\begin{aligned}
[\|w^n\|^2]_t=&\frac{\alpha}{4}((\tilde{u}^n+\tilde{u}^{n+1})(\tilde{v}^n+\tilde{v}^{n+1}),\ (u^n+u^{n+1})^2-(v^n+v^{n+1})^2)\\
&+\frac{\alpha}{4}((u^n+u^{n+1})(v^n+v^{n+1}),\ (\tilde{v}^n+\tilde{v}^{n+1})^2-(\tilde{u}^n+\tilde{u}^{n+1})^2)\\
&+\frac{\alpha}{8}(|\tilde{w}^n+\tilde{w}^{n+1}|^2,\ (u^n+u^{n+1})(\tilde{v}^n+\tilde{v}^{n+1})\\
&-(v^n+v^{n+1})(\tilde{u}^n+\tilde{u}^{n+1}))+(\tilde{u}^n+\tilde{u}^{n+1},\ \tilde{f}^n)+(\tilde{v}^n+\tilde{v}^{n+1},\ \tilde{g}^n).
\end{aligned}\tag{13}
$$

Assume

$$\|\tilde{\varphi}^n\|^2=\|\tilde{f}^n\|^2+\|\tilde{g}^n\|^2\leqslant M_0(k^{2p_1}+h^{2p_1})$$

and let M_i be a positive constant throughout, which may depend on $\|w^n\|_{L^\infty}$. Then from (8) it follows that the right side of (13) is bounded by

$$M_1(\|\tilde{w}^n\|^2+\|\tilde{w}^{n+1}\|^2+h^{-1}\|\tilde{w}^n\|^4+h^{-1}\|\tilde{w}^{n+1}\|^4+\|\tilde{\varphi}^n\|^2)$$

and so

$$(1-M_1k-M_1kh^{-1}\|\tilde{w}^{n+1}\|^2)\|\tilde{w}^{n+1}\|^2\leqslant(1+M_1k+M_1kh^{-1}\|\tilde{w}^n\|^2)\|\tilde{w}^n\|^2+k\|\tilde{\varphi}^n\|^2.\tag{14}$$

Inequality (14) is the basic error estimate for scheme (9).

We shall use the following lemma (see Zhu You-lan (1983)).

Lemma 1. *If $a>0$, $b>0$, $c>0$, $4ac<b^2$ and*

$$-az^2+bz-c\leqslant 0,\tag{15}$$

then

$$z\leqslant\frac{2c}{b}\quad\text{or}\quad z\geqslant\frac{b}{a}-\frac{2c}{b}.$$

Proof. Inequality (15) holds if and only if either $z\leqslant z_1$ or $z\geqslant z_2$, where

$$z_1=\frac{b-\sqrt{b^2-4ac}}{2a}$$

and

$$z_2=\frac{b+\sqrt{b^2-4ac}}{2a}.$$

Furthermore

$$z_1=\frac{b-b\sqrt{1-\frac{4ac}{b^2}}}{2a}<\frac{b-b\left(1-\frac{4ac}{b^2}\right)}{2a}=\frac{2c}{b}$$

and

$$z_2=\frac{b+b\sqrt{1-\frac{4ac}{b^2}}}{2a}>\frac{b+b\left(1-\frac{4ac}{b^2}\right)}{2a}=\frac{b}{a}-\frac{2c}{b}.$$

Thus the proof is completed.

Now let

$$Y^n=(1-M_1k-M_1kh^{-1}\|\tilde{w}^n\|^2)\|\tilde{w}^n\|^2.$$

Then from (14)

$$Y^{n+1}\leqslant\frac{1+M_1k+M_1kh^{-1}\|\tilde{w}^n\|^2}{1-M_1k-M_1kh^{-1}\|\tilde{w}^n\|^2}Y^n+k\|\tilde{\varphi}^n\|^2.\tag{16}$$

From Lemma 1 and (14) it follows that either

$$\|\widetilde{w}^{n+1}\|^2\leqslant\frac{2}{1-M_1k}\Big(\frac{1+M_1k+M_1kh^{-1}\|\widetilde{w}^n\|^2}{1-M_1k-M_1kh^{-1}\|\widetilde{w}^n\|^2}Y^n+k\|\widetilde{\varphi}^n\|^2\Big) \tag{17}$$

or

$$\|\widetilde{w}^{n+1}\|^2\geqslant\frac{h}{M_1k}(1-M_1k)-\frac{2}{1-M_1k}\Big(\frac{1+M_1k+M_1kh^{-1}\|\widetilde{w}^n\|^2}{1-M_1k-M_1kh^{-1}\|\widetilde{w}^n\|^2}Y^n+k\|\widetilde{\varphi}^n\|^2\Big). \tag{18}$$

Now let $p_1>\frac{1}{2}$, $p_2>\frac{1}{2}$ and k/h be suitably small. Then for sufficiently small h, $1-M_1k>\frac{1}{2}$.

We now use the induction to prove the error estimate. For $n=0$, we have $\|\widetilde{w}^0\|^2=0$ and $Y^0=0$. If (18) holds for $n=0$, then

$$\|\widetilde{w}^1\|^2\geqslant\frac{h}{2M_1k}-\frac{2kM_0}{1-M_1k}(k^{2p_1}+h^{2p_2})>2\|w_0\|^2$$

which is not possible and so from (17), we obtain

$$\|\widetilde{w}^1\|^2\leqslant 4M_0k(k^{2p_1}+h^{2p_2}).$$

Let M_2 be sufficiently large, and suppose that for all $l\leqslant n$,

$$\begin{cases} Y^l<\dfrac{M_0}{M_2}((1+M_2k)^l-1)(k^{2p_1}+h^{2p_2}), \\ \|w^l\|^2\leqslant\dfrac{10M_0}{M_2}\,e^{M_2T}(k^{2p_1}+h^{2p_2}). \end{cases}$$

Then from (16) we obtain

$$\begin{aligned} Y^{n+1}&\leqslant(1+M_2k)Y^n+M_0k(k^{2p_1}+h^{2p_2}) \\ &\leqslant\frac{M_0}{M_2}[(1+M_2k)^{n+1}-(1+M_2k)+M_2k](k^{2p_1}+h^{2p_2}) \\ &=\frac{M_0}{M_2}((1+M_2k)^{n+1}-1)(k^{2p_1}+h^{2p_2}). \end{aligned}$$

By (11) and

$$Y^{n+1}<\frac{M_0}{M_2}\,e^{M_2T}(k^{2p_1}+h^{2p_2}),\quad \text{for } nk\leqslant T,$$

inequality (18) cannot hold for $n+1$ and so from (17) it follows that

$$\begin{aligned} \|\widetilde{w}^{n+1}\|^2&\leqslant 10(Y^n+M_0k(k^{2p_1}+h^{2p_2}))\leqslant\frac{10M_0}{M_2}((1+M_2k)^n-1+M_2k)(k^{2p_1}+h^{2p_2}) \\ &=\frac{10M_0}{M_2}\,e^{M_2T}(k^{2p_1}+h^{2p_2}). \end{aligned}$$

Thus the induction is completed.

Theorem 1. *If $p_1>\frac{1}{2}$, $p_2>\frac{1}{2}$ and $\frac{k}{h}$ is suitably small, then for all $nk\leqslant T$,*

$$\|\widetilde{w}^n\|^2=O(k^{2p_1}+h^{2p_2}).$$

In particular, if $\frac{\partial^3 w}{\partial t^3}$ and $\frac{\partial^4 w}{\partial x^4}$ are continuous, then $p_1=p_2=2$ and so

$$\|\widetilde{w}^n\|^2=O(h^4).$$

Remark 1. If $p_1 > \frac{1}{2}$, $p_2 > \frac{1}{2}$ and $k = O(h^{\frac{1}{2}+\delta})$, $\delta > 0$, then the same result follows by using a technique similar to that of Zhu You-lan (1983).

Remark 2. Defour, Forten and Payne (1983) proposed the following scheme

$$\begin{cases} iW_t^n(x) + \frac{1}{2}(W_{x\bar{x}}^n(x) + W_{x\bar{x}}^{n+1}(x)) + \frac{\alpha}{4}(W^n(x) + W^{n+1}(x))(|W^n(x)|^2 \\ \quad + |W^{n+1}(x)|^2) = 0, \quad x \in R_h, \ n \geq 0, \\ W^0(x) = W_0(x), \quad x \in R_h. \end{cases}$$

Then not only (10) but also the following equality hold

$$\|W_x^n\|^2 - \frac{h}{2}\sum_{x\in R_h}|W^n(x)|^4 = \|W_x^0\|^2 - \frac{h}{2}\sum_{x\in R}|W^0(x)|^4.$$

The latter is a discrete analogy of (4). We can prove the same result as that of Theorem 1.

§ 3. Prediction-Correction Method

Let $\beta = \beta(h) \geq 0$ and $\overset{*}{W}{}^n(x) = \overset{*}{U}{}^n(x) + i\overset{*}{V}{}^n(x)$ be the prediction value of $W^{n+1}(x)$. Following Griffiths, Mitchell and Morris (1982), we consider the scheme

$$\begin{cases} \overset{*}{U}{}^n(x) = U^n(x) - \beta k V_{x\bar{x}}^n(x) - \alpha\beta k V^n(x)|W^n(x)|^2, \quad x \in R_h, \ n \geq 0, \\ \overset{*}{V}{}^n(x) = V^n(x) + \beta k U_{x\bar{x}}^n(x) + \alpha\beta k U^n(x)|W^n(x)|^2, \quad x \in R_h, \ n \geq 0, \\ U_t^n(x) + \frac{1}{2}(V_{x\bar{x}}^n(x) + V_{x\bar{x}}^{n+1}(x)) + \frac{\alpha}{8}(V^n(x) + \overset{*}{V}{}^n(x))|W^n(x) \\ \quad + \overset{*}{W}{}^n(x)|^2 = 0, \quad x \in R_h, \ n \geq 0, \\ V_t^n(x) - \frac{1}{2}(U_{x\bar{x}}^n(x) + U_{x\bar{x}}^{n+1}(x)) - \frac{\alpha}{8}(U^n(x) + \overset{*}{U}{}^n(x))|W^n(x) \\ \quad + \overset{*}{W}{}^n(x)|^2 = 0, \quad x \in R_h, \ n \geq 0, \\ U^0(x) = u_0(x), \quad x \in R_h, \\ V^0(x) = v_0(x), \quad x \in R_h. \end{cases} \tag{19}$$

By substituting the first two formulas into the next two of (19), we obtain

$$\begin{cases} U_t^n(x) + \frac{1}{2}(V^n(x) + V^{n+1}(x)) + \frac{\alpha}{4}V^n(x)B(U^n(x), V^n(x)) \\ \quad + A_1(U^n(x), V^n(x)) = 0, \quad x \in R_h, \ n \geq 0, \\ V_t^n(x) - \frac{1}{2}(U^n(x) + U^{n+1}(x)) - \frac{\alpha}{4}U^n(x)B(U^n(x), V^n(x)) \\ \quad + A_2(U^n(x), V^n(x)) = 0, \quad x \in R_h, \ n \geq 0, \end{cases} \tag{20}$$

where

$$A_1(U^n(x), V^n(x)) = \frac{\alpha\beta k}{8}(U_{x\bar{x}}^n(x) + \alpha U^n(x)|W^n(x)|^2)B(U^n(x), V^n(x)),$$

$$A_2(U^n(x), V^n(x)) = \frac{\alpha\beta k}{8}(V_{x\bar{x}}^n(x) + \alpha V^n(x)|W^n(x)|^2)B(U^n(x), V^n(x)),$$

$$B(U^n(x), V^n(x)) = (2U^n(x) - \beta k V_{x\bar{x}}^n(x) - \alpha\beta k V^n(x)|W^n(x)|^2)^2 \\ \qquad + (2V^n(x) + \beta k U_{x\bar{x}}^n(x) + \alpha\beta k U^n(x)|W^n(x)|^2)^2.$$

Let $\tilde{w}^n(x)$, $\tilde{f}^n(x)$ and $\tilde{g}^n(x)$ be the same as before, then

$$\begin{aligned}&\tilde{u}_t^n(x)+\frac{1}{2}(\tilde{v}^n(x)+\tilde{v}^{n+1}(x))+\frac{\alpha}{4}\tilde{v}^n(x)B(u^n(x)\\&\quad+\tilde{u}^n(x),\ v^n(x)+\tilde{v}^n(x))+G_5^n(x)=\tilde{f}^n(x),\\&\tilde{v}_t^n(x)-\frac{1}{2}(\tilde{u}^n(x)+\tilde{u}^{n+1}(x))-\frac{\alpha}{4}\tilde{u}^n(x)B(u^n(x)\\&\quad+\tilde{u}^n(x),\ v^n(x)+\tilde{v}^n(x))+G_6^n(x)=\tilde{g}^n(x),\end{aligned}\tag{21}$$

where

$$\begin{aligned}G_5^n(x)=&\frac{\alpha\beta k}{8}[\tilde{u}_{xx}^n(x)+\alpha\tilde{u}^n(x)|w(x)+\tilde{w}^n(x)|^2+\alpha u^n(x)(|\tilde{w}^n(x)|^2\\&+2u^n(x)\tilde{u}^n(x)+2v^n(x)\tilde{v}^n(x))]B(u^n(x)+\tilde{u}^n(x),\ v^n(x)+\tilde{v}^n(x))\\&+\frac{\alpha}{8}(2v^n(x)+\beta ku_{xx}^n(x)+\alpha\beta ku^n(x)|w^n(x)|^2)(B(u^n(x)\\&+\tilde{u}^n(x),\ v^n(x)+\tilde{v}^n(x))-B(u^n(x),\ v^n(x))),\\G_6^n(x)=&\frac{\alpha\beta k}{8}[\tilde{v}_{xx}^n(x)+\alpha\tilde{v}^n(x)|w^n(x)+\tilde{w}^n(x)|^2+\alpha v^n(x)(|\tilde{w}^n(x)|^2\\&+2u^n(x)\tilde{u}^n(x)+2v^n(x)\tilde{v}^n(x))]B(u^n(x)+\tilde{u}^n(x),\ v^n(x)+\tilde{v}^n(x))\\&-\frac{\alpha}{8}(2u^n(x)-\beta kv_{xx}^n(x)-\alpha\beta kv^n(x)|w^n(x)|^2)(B(u^n(x)\\&+\tilde{u}^n(x),\ v^n(x)+\tilde{v}^n(x))-B(u^n(x),\ v^n(x))).\end{aligned}$$

By taking the scalar products of the first two equations of (21) with $\tilde{u}^n(x)+\tilde{u}^{n+1}(x)$ and $\tilde{v}^n(x)+\tilde{v}^{n+1}(x)$ respectively and summing up the two results, then from (5) and (6) we have

$$\begin{aligned}&[\|\tilde{w}^n\|^2]_t+2(\tilde{u}^n,\ G_5^n)+2(\tilde{v}^n,\ G_6^n)+k(\tilde{u}_t^n,\ G_5^n)+k(\tilde{u}_t^n,\ G_6^n)\\&\quad=(\tilde{u}^n+\tilde{u}^{n+1},\ \tilde{f}^n)+(\tilde{v}^n+\tilde{v}^{n+1},\ \tilde{g}^n).\end{aligned}\tag{22}$$

Assume $k=O(h^2)$. Then from (8) it follows that

$$|2(\tilde{u}^n,\ G_5^n)+2(\tilde{v}^n,\ G_6^n)|\le M_3\sum_{r=0}^{4}h^{-r}\|\tilde{w}^n\|^{2r+2}\tag{23}$$

and

$$|k(\tilde{u}_t^n,\ G_5^n)+k(\tilde{v}_t^n,\ G_6^n)|\le sk^2\|\tilde{u}_t^n\|^2+\frac{M_4}{s}\sum_{r=0}^{8}h^{-r}\|\tilde{w}^n\|^{2r+2}.\tag{24}$$

By substituting (23) and (24) into (22), we obtain

$$[\|\tilde{w}^n\|^2]_t\le sk^2\|\tilde{w}_t^n\|^2+M_5\Big(1+\frac{1}{s}\Big)\Big(\sum_{r=0}^{8}h^{-r}\|\tilde{w}^n\|^{2r+2}+\|\tilde{\varphi}^n\|^2\Big)\tag{25}$$

and so

$$\|\tilde{w}^{n+1}\|^2\le M_6\,\frac{(1+s)(1+2sk)}{s(1-2sk)}\Big(\sum_{r=0}^{8}kh^{-r}\|\tilde{w}^n\|^{2r+2}+k\|\tilde{\varphi}^n\|^2\Big).$$

Let s be suitably small such that $\dfrac{(1+s)(1+2sk)}{s(1-2sk)}\le C_0$ and

$$\rho^n=C_0M_6k\sum_{l=0}^{n-1}\|\tilde{\varphi}^l\|^2.$$

Then

$$\|\tilde{w}^n\|^2\le\rho^n+M_7k\sum_{l=0}^{n-1}\sum_{r=0}^{8}h^{-r}\|\tilde{w}^l\|^{2r+2},\tag{26}$$

Inequality (26) is the basic error estimate for scheme (19).

Lemma 2 (see Kuo Pen-yu (1979)). *If the following conditions are satisfied*

(i) D^n *is a non-negative function,* h, C_1 *and* C_2 *are positive constants,*

(ii) $D^0 \leqslant C_1$ *and*

$$D^n \leqslant C_1 + C_2 k \sum_{l=0}^{n-1} \sum_{r=0}^{3} h^{-r} (D^l)^{r+1},$$

(iii) $C_1 e^{5C_2 T} \leqslant 1$,

then for all $nk \leqslant T$,

$$D^n \leqslant C_1 e^{5C_2 nk}.$$

Now by putting $D^n = \|\tilde{w}^n\|^2$ and $C_1 = \rho^{[\frac{T}{k}]+1}$ in Lemma 2, we get the following result:

Theorem 2. *If* $p_1 > \frac{1}{4}$, $p_2 > \frac{1}{2}$, $k = O(h^2)$, *then for all* $nk \leqslant T$,

$$\|\tilde{w}^n\|^2 = O(k^{2p_1} + h^{2p_2}).$$

In particular if $\frac{\partial^2 U}{\partial t^2}$ *and* $\frac{\partial^4 U}{\partial x^4}$ *are continuous, then* $p_1 = 1$, $p_2 = 2$ *and so*

$$\|\tilde{w}^n\|^2 = O(h^4).$$

Remark 3. If $\beta = 0$, we can prove that

$$\|W^{n+1}\|^2 = \|W^n\|^2 + k\|W_t^n\|^2.$$

Thus if $\|W_t^n\|^2$ is large, then $\|W^n\|^2$ increases quickly and so the computation is not stable for a long time.

Remark 4. If $\beta = 1 + O(k)$, then $\|\tilde{\varphi}^n\| = O(k^2 + h^2)$ and so scheme (19) is of high order accuracy.

Remark 5. Another prediction-correction scheme is

$$\begin{cases} \overset{*}{U}{}^n(x) = U^n(x) - \beta k V_{\bar{x}x}^n(x) - \alpha \beta k V^n(x) |W^n(x)|^2, & x \in R_h,\ n \geqslant 0, \\ \overset{*}{V}{}^n(x) = V^n(x) + \beta k V_{\bar{x}x}^n(x) + \alpha \beta k U^n(x) |W^n(x)|^2, & x \in R_h,\ n \geqslant 0, \\ U_t^n(x) + \frac{1}{2}(V_{\bar{x}x}^n(x) + V_{\bar{x}x}^{n+1}(x)) + \frac{\alpha}{8}(V^n(x) + V^{n+1}(x)) |W^n(x) & \\ \quad + \overset{*}{W}{}^n(x)^2| = 0, & x \in R_h,\ n \geqslant 0, \\ V_t^n(x) - \frac{1}{2}(U_{\bar{x}x}^n(x) + U_{\bar{x}x}^{n+1}(x)) - \frac{\alpha}{8}(U^n(x) + U^{n+1}(x)) |W^n(x) & \\ \quad + \overset{*}{W}{}^n(x)|^2 = 0, & x \in R_h,\ n \geqslant 0, \end{cases}$$

whose solution satisfies equality (10). The same result as that of Theorem 2 can be proved.

§ 4. Galerkin Method

In this section we consider the generalized solution of (1) which is the function $w(x, t) \in L^\infty(0, T; L^2(\mathbb{R}))$ satisfying

$$\begin{cases} i\int_{\mathbb{R}} \frac{\partial w}{\partial t} \psi\, dx - \int_{\mathbb{R}} \frac{\partial w}{\partial x} \frac{\partial \psi}{\partial x} dx + \alpha \int_{\mathbb{R}} w|w|^2 \psi\, dx = 0, & \forall \psi \in H^1(R),\ 0 < t \leqslant T, \\ U(x, 0) = w_0(x), & x \in \mathbb{R}. \end{cases}$$

We use the product-approximation Galerkin method. Let S_h be the subspace of $H^1(\mathbf{R})$ and $W(x, t)$ be the approximation of $w(x, t)$, where

$$W(x, t) = \sum_{j=-\infty}^{\infty} W_j(t)\psi_j(x),$$

$\psi_j(x)$ are the trail functions of S_h and $W(x, t)$ satisfies

$$i\int_{\mathbf{R}} \frac{\partial W}{\partial t}\psi_j\,dx - \int_{\mathbf{R}} \frac{\partial W}{\partial x}\frac{\partial \psi_j}{\partial x}\,dx + \alpha\sum_{l=-\infty}^{\infty}\int_{\mathbf{R}} W_l(t)\,|W_l(t)|^2\psi_l\psi_j\,dx = 0, \quad \forall \psi_j \in S_h.$$

If we take $\psi_j(x)$ to be the hat function

$$\psi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h}, & x_{j-1} \leqslant x < x_j, \\ \dfrac{x_{j+1} - x}{h}, & x_j \leqslant x \leqslant x_{j+1}, \end{cases}$$

then we have

$$\frac{i}{6}\left(\frac{dW_{j-1}(t)}{dt} + 4\frac{dW_j(t)}{dt} + \frac{dW_{j+1}(t)}{dt}\right) + W_{x\bar{x},j}(t)$$
$$+\frac{\alpha}{6}(W_{j-1}(t)\,|W_{j-1}(t)|^2 + 4W_j(t)\,|W_j(t)|^2 + W_{j+1}(t)\,|W_{j+1}(t)|^2) = 0. \tag{27}$$

If we use the Crank-Nicolson method to discretize the variable t, then we obtain

$$\frac{i}{6}(W^n_{t,j-1} + 4W^n_{t,j} + W^n_{t,j-1}) + \frac{1}{2}(W^n_{x\bar{x},j} + W^{n+1}_{x\bar{x},j})$$
$$+\frac{\alpha}{48}[(W^n_{j-1} + W^{n+1}_{j-1})\,|W^n_{j-1} + W^{n+1}_{j-1}|^2 + 4(W^n_j + W^{n+1}_j)\,|W^n_j + W^{n+1}_j|^2$$
$$+(W^n_{j+1} + W^{n+1}_{j+1})\,|W^n_{j+1} + W^{n+1}_{j+1}|^2] = 0. \tag{28}$$

Let $\mathscr{M}$ and $\mathscr{D}$ be the operators defined as

$$\mathscr{M}\eta^n_j = \frac{1}{6}(\eta^n_{j-1} + 4\eta^n_j + \eta^n_{j+1}), \quad \mathscr{D}\eta^n_j = \eta^n_{x\bar{x},j}.$$

Then $\mathscr{M}$ is a positive operator and $\mathscr{M}^{-1}$ exists. Scheme (28) is identical to the following equation

$$\frac{i}{6}W^n_{t,j} + \frac{\mathscr{M}^{-1}\mathscr{D}}{2}(W^n_j + W^{n+1}_j) + \frac{\alpha}{8}(W^n_j + W^{n+1}_j)\,|W^n_j + W^{n+1}_j|^2 = 0.$$

Since $\mathscr{M}^{-1}\mathscr{D} = \mathscr{D}\mathscr{M}^{-1}$, by an argument similar to that in Section 2 we get the convergence for scheme (28).

Griffiths, Mitchell and Morris (1982) proposed the following prediction correction method

$$\begin{cases} \dfrac{i}{6}(\mathring{W}^n_{j-1} + 4\mathring{W}^n_j + \mathring{W}^n_{j+1}) = \dfrac{i}{6}(W^n_{j-1} + 4W^n_j + W^n_{j-1}) - \beta k W^n_{x\bar{x},j} \\ \qquad\qquad -\dfrac{\alpha\beta k}{6}(W^n_{j-1}|W^n_{j-1}|^2 + 4W^n_j|W^n_j|^2 + W^n_{j+1}|W^n_{j+1}|^2), \\ \dfrac{i}{6}(w^n_{t,j-1} + 4w^n_{t,j} + w^n_{t,j+1}) + \dfrac{1}{2}(W^n_{x\bar{x},j} + W^{n+1}_{x\bar{x},j}) \\ \quad +\dfrac{\alpha}{48}((W^n_{j-1} + \mathring{W}^n_{j-1})\,|W^n_{j-1} + \mathring{W}^n_{j-1}|^2 + 4(W^n_j + \mathring{W}^n_j)\,|W^n_j + \mathring{W}^n_j|^2 \\ \quad +(W^n_{j+1} + \mathring{W}^n_{j+1})\,|W^n_{j+1} + \mathring{W}^n_{j+1}|^2 = 0. \end{cases} \tag{29}$$

The same convergence as that in Section 3 can be proved. The numerical result showed that (29) is stable and has high order accuracy provided $\beta=1+O(k)$ (see Griffiths, Mitchell, Morris (1982)).

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