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A SYSTEM OF PLANE ELASTICITY CANONICAL INTEGRAL EQUATIONS AND ITS APPLICATION*

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Abstract

In this paper, we obtain a new system of canonical integral equations for the plane elasticity problem over an exterior circular domain, and give its numerical solution. Coupling with the classical finite element method, it can be used for solving general plane elasticity exterior boundary value problems. This system of highly singular integral equations is also an exact boundary condition on the artificial boundary. It can be approximated by a series of nonsingular integral boundary conditions.

§ 1. Introduction

The canonical boundary reduction, suggested by Feng Kang[1], can be applied to the plane elasticity problem[2,4]. Let Ω be a domain with smooth boundary Γ. Taking displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in directions x_1 and x_2 as basic unknown functions, we have the plane elasticity equations with traction boundary conditions as follows

$$\begin{cases} (\lambda+2\mu) \operatorname{grad} \operatorname{div} \boldsymbol{u}-\mu \operatorname{rot} \operatorname{rot} \boldsymbol{u}=0, & \text{in } \Omega, \\ \sum\limits_{j=1}^{2} \sigma_{ij} n_j = g_i, \quad i=1, 2, & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\boldsymbol{u}=u_1\boldsymbol{e}_1+u_2\boldsymbol{e}_2$, σ_{ij} $(i, j=1, 2)$ are components of stress, λ and μ are Lamé coefficients, $\lambda>0$, $\mu>0$, and (n_1, n_2) are the outward normal direction cosines on Γ.

Let

$$\mathscr{R}=\{\boldsymbol{v}\in H^1(\Omega)^2 \mid \boldsymbol{v}=(c_1-c_3x_2, c_2+c_3x_1)^T, c_1, c_2, c_3\in \mathbf{R}\};$$

then problem (1) has one and only one solution in $H^1(\Omega)^2/\mathscr{R}$ when $\boldsymbol{g}$ satisfies the consistency conditions

$$\int_\Gamma g_i ds=0,\ i=1, 2 \quad \text{and} \quad \int_\Gamma (x_1g_2-x_2g_1)ds=0.$$

From [4] we know that the boundary value problem (1) is equivalent to the canonical integral equation on Γ:

$$\boldsymbol{g}(s)=\int_\Gamma K(s, s')\boldsymbol{u}_0(s')ds'. \tag{2}$$

Particularly, by using the separation of variables, [4] has given a system of canonical integral equations with respect to exterior circular domain:

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$$\begin{bmatrix} g_r(\theta) \\ g_\theta(\theta) \end{bmatrix} = \begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix} * \begin{bmatrix} u_r(R, \theta) \\ u_\theta(R, \theta) \end{bmatrix}, \tag{3}$$

where

$$K_{rr} = K_{\theta\theta} = -\frac{ab}{(a+b)2\pi R \sin^2\theta/2} + \frac{2b^2}{(a+b)R}\delta(\theta) + \frac{ab}{\pi R(a+b)},$$

$$K_{r\theta} = -K_{\theta r} = -\frac{ab}{(a+b)\pi R}\operatorname{ctg}\frac{\theta}{2} + \frac{2b^2}{(a+b)R}\delta'(\theta),$$

$$\boldsymbol{g} = g_r\boldsymbol{e}_r + g_\theta\boldsymbol{e}_\theta, \quad \boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta, \quad a = \lambda + 2\mu, \quad b = \mu,$$

$*$ denotes the convolution, and R is the radius of the circle.

When Ω is the exterior to an arbitrary smooth closed curve Γ, the solution of problem (1) must satisfy the condition at infinity, i.e. u_1 and u_2 are bounded at infinity. Then problem (1) has one and only one solution in $W_0^1(\Omega)^2/\mathscr{R}$, where

$$W_0^1(\Omega) = \left\{\frac{u}{\sqrt{1+r^2}\ln(2+r^2)} \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), \ i=1, 2, \ r = \sqrt{x_1^2 + x_2^2}\right\},$$

$$\mathscr{R} = \{\boldsymbol{v} \in W_0^1(\Omega)^2 \mid \boldsymbol{v} = (c_1, c_2)^T, \ c_1, c_2 \in \boldsymbol{R}\},$$

when $\boldsymbol{g}$ satisfies the consistency conditions:

$$\int_\Gamma g_i ds = 0, \quad i = 1, 2.$$

We can draw in Ω a circle Γ' with radius R. Then the original problem reduces to a new boundary value problem over a bounded domain Ω_1; its boundary condition on the artificial boundary Γ' is just the system of canonical integral equations (3).

Of course the boundary condition on Γ' is not unique. Recently [5] gives another integral boundary condition. Noticing that

$$\begin{cases} \sum_{n=1}^\infty \frac{1}{n}\cos n\theta = -\ln\left|2\sin\frac{\theta}{2}\right|, \\ \sum_{n=1}^\infty n\cos n\theta = -\frac{1}{4\sin^2\theta/2}, \\ \sum_{n=1}^\infty n\sin n\theta = -\pi\delta'(\theta), \end{cases} \tag{4}$$

we can write that condition as

$$\begin{bmatrix} -\frac{\partial u_1}{\partial r} + p\cos\theta \\ -\frac{\partial u_2}{\partial r} + p\sin\theta \end{bmatrix}_{\Gamma'}$$

$$= \frac{1}{R}\begin{bmatrix} \frac{2a}{a+b}\left(-\frac{1}{4\pi\sin^2\theta/2}\right) & -\frac{a-b}{a+b}\delta'(\theta) \\ \frac{a-b}{a+b}\delta'(\theta) & \frac{2a}{a+b}\left(-\frac{1}{4\pi\sin^2\theta/2}\right) \end{bmatrix} * \begin{bmatrix} u_1(R, \theta) \\ u_2(R, \theta) \end{bmatrix}, \tag{5}$$

where $p = -\frac{\lambda+\mu}{\mu}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)$. Obviously, it is different from (3) in that the left side of (5) is not the tractions on the artificial boundary Γ'.

Below we shall apply the method of complex analysis and obtain a new integral boundary condition on Γ'. It is a system of canonical integral equations different from (3), and represents the relationship between tractions (g_1, g_2) and displacements (u_1, u_2) on Γ'.

§ 2. A New System of Canonical Integral Equations

From the mathematical theory of elasticity[9], we know that the solution of a plane elasticity problem can be represented by two analytic functions $\phi(z)$ and $\psi(z)$:

$$\begin{cases} u_1=\dfrac{1}{2\mu}\mathrm{Re}\left[\dfrac{\lambda+3\mu}{\lambda+\mu}\phi(z)-\bar{z}\phi'(z)-\psi'(z)\right], \\ u_2=\dfrac{1}{2\mu}\mathrm{Im}\left[\dfrac{\lambda+3\mu}{\lambda+\mu}\phi(z)+\bar{z}\phi'(z)+\psi'(z)\right], \\ \sigma_x=\mathrm{Re}[2\phi'(z)-\bar{z}\phi''(z)-\psi''(z)], \\ \sigma_y=\mathrm{Re}[2\phi'(z)+\bar{z}\phi''(z)+\psi''(z)], \\ \sigma_{xy}=\mathrm{Im}[\bar{z}\phi''(z)+\psi''(z)]. \end{cases} \tag{6}$$

Let Ω be the exterior domain to the unit circle, $z=re^{i\theta}$,

$$F(z)=\frac{\lambda+3\mu}{\lambda+\mu}\phi(z)-\frac{1}{z}\phi'(z)-\psi'(z),$$

$$G(z)=\frac{\lambda+3\mu}{\lambda+\mu}\phi(z)+\frac{1}{z}\phi'(z)+\psi'(z).$$

Then from (6) we have

$$u_1(1,\theta)=\frac{1}{2\mu}\mathrm{Re}\,F(z)|_{r=1},\quad u_2(1,\theta)=\frac{1}{2\mu}\mathrm{Im}\,G(z)|_{r=1},$$

and

$$\begin{cases} \sigma_x(1,\theta)=\mathrm{Re}[2\phi'(e^{i\theta})-e^{-i\theta}\phi''(e^{i\theta})-\psi''(e^{i\theta})], \\ \sigma_y(1,\theta)=\mathrm{Re}[2\phi'(e^{i\theta})+e^{-i\theta}\phi''(e^{i\theta})+\psi''(e^{i\theta})], \\ \sigma_{xy}(1,\theta)=\mathrm{Im}[e^{-i\theta}\phi''(e^{i\theta})+\psi''(e^{i\theta})]. \end{cases} \tag{7}$$

Because $F(z)$ and $G(z)$ are analytic over Ω, we can apply the harmonic canonical integral equation[1,2], and obtain

$$\begin{aligned} K(\theta)*u_1(1,\theta)&=-\frac{\partial}{\partial r}\left[\frac{1}{2\mu}\mathrm{Re}\,F(z)\right]\Big|_{r=1} \\ &=-\frac{1}{2\mu}\mathrm{Re}\left[\frac{\lambda+3\mu}{\lambda+\mu}e^{i\theta}\phi'(e^{i\theta})+e^{-i\theta}\phi'(e^{i\theta})-\phi''(e^{i\theta})-e^{i\theta}\psi''(e^{i\theta})\right], \\ K(\theta)*u_2(1,\theta)&=-\frac{\partial}{\partial r}\left[\frac{1}{2\mu}\mathrm{Im}\,G(z)\right]\Big|_{r=1} \\ &=-\frac{1}{2\mu}\mathrm{Im}\left[\frac{\lambda+3\mu}{\lambda+\mu}e^{i\theta}\phi'(e^{i\theta})-e^{-i\theta}\phi'(e^{i\theta})+\phi''(e^{i\theta})+e^{i\theta}\psi''(e^{i\theta})\right], \end{aligned}$$

where

$$K(\theta)=-\frac{1}{4\pi\sin^2\theta/2}.$$

Moreover,

$$\frac{d}{d\theta}u_1(1,\theta)=-\frac{1}{2\mu}\mathrm{Im}\left[\frac{\lambda+3\mu}{\lambda+\mu}e^{i\theta}\phi'(e^{i\theta})+e^{-i\theta}\phi'(e^{i\theta})-\phi''(e^{i\theta})-e^{i\theta}\psi''(e^{i\theta})\right],$$

$$\frac{d}{d\theta}u_2(1,\theta)=\frac{1}{2\mu}\mathrm{Re}\Big[\frac{\lambda+3\mu}{\lambda+\mu}e^{i\theta}\phi'(e^{i\theta})-e^{-i\theta}\phi'(e^{i\theta})+\phi''(e^{i\theta})+e^{i\theta}\psi''(e^{i\theta})\Big].$$

From the above equalities we can get

$$\mathrm{Re}\,\phi'(e^{i\theta})=\frac{\mu(\lambda+\mu)}{\lambda+3\mu}\Big\{-\cos\theta\,[K(\theta)*u_1(1,\theta)]-\sin\theta\frac{d}{d\theta}u_1(1,\theta)$$
$$-\sin\theta[K(\theta)*u_2(1,\theta)]+\cos\theta\frac{d}{d\theta}u_2(1,\theta)\Big\},$$

$$\mathrm{Im}\,\phi'(e^{i\theta})=\frac{\mu(\lambda+\mu)}{\lambda+3\mu}\Big\{\sin\theta[K(\theta)*u_1(1,\theta)]-\cos\theta\frac{d}{d\theta}u_1(1,\theta)$$
$$-\cos\theta\,[K(\theta)*u_2(1,\theta)]-\sin\theta\frac{d}{d\theta}u_2(1,\theta)\Big\},$$

$$\mathrm{Re}[e^{-i\theta}\phi''(e^{i\theta})+\psi''(e^{i\theta})]=\mu\Big\{\cos\theta[K(\theta)*u_1(1,\theta)]+\sin\theta\frac{d}{d\theta}u_1(1,\theta)$$
$$-\sin\theta[K(\theta)*u_2(1,\theta)]+\cos\theta\frac{d}{d\theta}u_2(1,\theta)\Big\}$$
$$+\cos 2\theta\,\mathrm{Re}\,\phi'(e^{i\theta})+\sin 2\theta\,\mathrm{Im}\,\phi'(e^{i\theta}),$$

$$\mathrm{Im}\,[e^{-i\theta}\phi''(e^{i\theta})+\psi''(e^{i\theta})]=\mu\Big\{-\sin\theta[K(\theta)*u_1(1,\theta)]+\cos\theta\frac{d}{d\theta}u_1(1,\theta)$$
$$-\cos\theta[K(\theta)*u_2(1,\theta)]-\sin\theta\frac{d}{d\theta}u_2(1,\theta)\Big\}$$
$$-\sin 2\theta\,\mathrm{Re}\,\phi'(e^{i\theta})+\cos 2\theta\,\mathrm{Im}\,\phi'(e^{i\theta}).$$

Substituting these results into (7), according to the relation

$$\begin{bmatrix}g_1(\theta)\\ g_2(\theta)\end{bmatrix}=\begin{bmatrix}\sigma_x & \sigma_{xy}\\ \sigma_{xy} & \sigma_y\end{bmatrix}\begin{bmatrix}-\cos\theta\\ -\sin\theta\end{bmatrix}_{r=1}=\begin{bmatrix}-(\sigma_x\cos\theta+\sigma_{xy}\sin\theta)\\ -(\sigma_{xy}\cos\theta+\sigma_y\sin\theta)\end{bmatrix}_{r=1},$$

we can obtain a system of canonical integral equations:

$$\begin{bmatrix}g_1(\theta)\\ g_2(\theta)\end{bmatrix}=\begin{bmatrix}\frac{2ab}{a+b}\Big(-\frac{1}{4\pi\sin^2\theta/2}\Big) & \frac{2b^2}{a+b}\delta'(\theta)\\ -\frac{2b^2}{a+b}\delta'(\theta) & \frac{2ab}{a+b}\Big(-\frac{1}{4\pi\sin^2\theta/2}\Big)\end{bmatrix}*\begin{bmatrix}u_1(1,\theta)\\ u_2(1,\theta)\end{bmatrix}.$$

If Ω is the exterior domain to the circle with radius R, we have

$$\begin{bmatrix}g_1(\theta)\\ g_2(\theta)\end{bmatrix}=\frac{1}{R}\begin{bmatrix}-\frac{2ab}{a+b}\frac{1}{4\pi\sin^2\theta/2} & \frac{2b^2}{a+b}\delta'(\theta)\\ -\frac{2b^2}{a+b}\delta'(\theta) & -\frac{2ab}{a+b}\frac{1}{4\pi\sin^2\theta/2}\end{bmatrix}*\begin{bmatrix}u_1(R,\theta)\\ u_2(R,\theta)\end{bmatrix}. \qquad (8)$$

It has a simpler form than (3).

Now we write (8) as

$$g=\mathscr{K}u_0,$$

and define a bilinear form

$$\hat{D}(u_0,v_0)=\int_\Gamma v_0\cdot\mathscr{K}u_0\,ds.$$

Lemma 1. *$\hat{D}(u_0,v_0)$ is a positive definite symmetric continuous and V-elliptic bilinear form on $V(\Gamma)=H^{\frac{1}{2}}(\Gamma)^2/\mathscr{R}(\Gamma)$, where*

$$\mathscr{R}(\Gamma)=\{[c_1,\ c_2]\,|\,c_1 \text{ and } c_2 \text{ are constants}\}.$$

Proof. Let

$$u_0(\theta)=\left[\sum_{-\infty}^{\infty} a_n e^{in\theta},\ \sum_{-\infty}^{\infty} b_n e^{in\theta}\right]^T,\quad a_{-n}=\bar{a}_n,\ b_{-n}=\bar{b}_n,$$

$$v_0(\theta)=\left[\sum_{-\infty}^{\infty} c_n e^{in\theta},\ \sum_{-\infty}^{\infty} d_n e^{in\theta}\right]^T,\quad c_{-n}=\bar{c}_n,\ d_{-n}=\bar{d}_n,\quad n=0,\ 1,\ \cdots.$$

Then

$$\hat{D}(u_0,\ v_0)=2\pi\sum_{-\infty}^{\infty}[\bar{c}_n,\ \ \bar{d}_n]\begin{bmatrix}\dfrac{2ab}{a+b}|n| & \dfrac{2b^2}{a+b}in\\ -\dfrac{2b^2}{a+b}in & \dfrac{2ab}{a+b}|n|\end{bmatrix}\begin{bmatrix}a_n\\ b_n\end{bmatrix}.$$

It is easily seen that $\hat{D}(u_0,\ v_0)=\hat{D}(v_0,\ u_0)$. Particularly, if $v_0=u_0$, we have

$$\begin{aligned}\hat{D}(u_0,\ u_0)&=2\pi\sum_{-\infty}^{\infty}\left\{\frac{2ab}{a+b}|n|(|a_n|^2+|b_n|^2)+\frac{2b^2}{a+b}in(\bar{a}_n b_n-\bar{b}_n a_n)\right\}\\&=2\pi\sum_{-\infty}^{\infty}\left\{\frac{2ab}{a+b}|n|\left|a_n+i(\operatorname{sign} n)\frac{b}{a}b_n\right|^2+\frac{2b(a-b)}{a}|n|\,|b_n|^2\right\}\geqslant 0;\end{aligned}$$

the equality holds if and only if $u_0\in\mathscr{R}(\Gamma)$. Moreover,

$$\begin{aligned}|\hat{D}(u_0,\ v_0)|&=\left|2\pi\sum_{-\infty}^{\infty}\left\{\frac{2ab}{a+b}|n|(a_n\bar{c}_n+b_n\bar{d}_n)+\frac{2b^2}{a+b}in(b_n\bar{c}_n-a_n\bar{d}_n)\right\}\right|\\&\leqslant 2\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\left\{\frac{2b}{a+b}\sqrt{n^2+1}\,[a(|a_n|\,|c_n|+|b_n|\,|d_n|)+b(|b_n|\,|c_n|+|a_n|\,|d_n|)]\right\}\\&\leqslant 2b\cdot 2\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\sqrt{n^2+1}\,(|a_n|^2+|b_n|^2)^{\frac12}(|c_n|^2+|d_n|^2)^{\frac12}\\&\leqslant 2b\|u_0\|_{V(\Gamma)}\|v_0\|_{V(\Gamma)},\end{aligned}$$

where

$$\|u_0\|_{V(\Gamma)}=\left[2\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\sqrt{n^2+1}\,(|a_n|^2+|b_n|^2)\right]^{\frac12},$$

and

$$\begin{aligned}\hat{D}(u_0,\ u_0)&=2\pi\sum_{-\infty}^{\infty}\left\{\frac{2b^2}{a+b}|n|\,|a_n+i(\operatorname{sign} n)b_n|^2+\frac{2b(a-b)}{a+b}|n|(|a_n|^2+|b_n|^2)\right\}\\&\geqslant 2\pi\sum_{-\infty}^{\infty}\frac{2b(a-b)}{a+b}|n|(|a_n|^2+|b_n|^2)\\&\geqslant\frac{\sqrt{2}\,b(a-b)}{a+b}\,2\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\sqrt{n^2+1}\,(|a_n|^2+|b_n|^2)\\&=\frac{\sqrt{2}\,b(a-b)}{a+b}\|u_0\|^2_{V(\Gamma)}.\end{aligned}$$

Then

$$\frac{\sqrt{2}\,b(a-b)}{a+b}\|u_0\|^2_{V(\Gamma)}\leqslant\hat{D}(u_0,\ u_0)\leqslant 2b\|u_0\|^2_{V(\Gamma)},\tag{9}$$

which implies that $\|u_0\|_{\hat{D}}=[\hat{D}(u_0,\ u_0)]^{\frac12}$ defines an equivalent norm on $V(\Gamma)$. The proof is complete.

From Lemma 1 and the Lax-Milgram lemma[10] we immediately obtain

Theorem 1. *If the boundary traction $g\in H^{-\frac{1}{2}}(\Gamma)^2$ satisfies compatibility conditions $\int_\Gamma g_i ds=0$ $(i=1, 2)$, then the variational problem*

$$\begin{cases} \text{Find } u_0\in H^{\frac{1}{2}}(\Gamma)^2 \quad \text{such that} \\ \hat{D}(u_0, v_0)=\hat{F}(v_0), \quad \forall v_0\in H^{\frac{1}{2}}(\Gamma)^2 \end{cases} \tag{10}$$

has a unique solution up to a vector in $\mathscr{R}(\Gamma)$, and the solution depends on the given traction continuously:

$$\|u_0\|_{V(\Gamma)}\leqslant\frac{a+b}{\sqrt{2}\,b(a-b)}\|g\|_{H^{-\frac{1}{2}}(\Gamma)^2}, \tag{11}$$

where $\hat{F}(v_0)=\int_\Gamma g\cdot v_0\,ds.$

In addition, by taking the coefficients of the Fourier expansion of (8), we can obtain its inverse formula:

$$\begin{bmatrix} u_1(R, \theta) \\ u_2(R, \theta)\end{bmatrix}=R\begin{bmatrix} -\frac{a}{2b(a-b)\pi}\ln\left|2\sin\frac{\theta}{2}\right| & \frac{1}{4(a-b)\pi}[\pi\operatorname{sign}\theta-\theta] \\ -\frac{1}{4(a-b)\pi}[\pi\operatorname{sign}\theta-\theta] & -\frac{a}{2b(a-b)\pi}\ln\left|2\sin\frac{\theta}{2}\right|\end{bmatrix}$$
$$*\begin{bmatrix} g_1(\theta) \\ g_2(\theta)\end{bmatrix}+\begin{bmatrix} C_1 \\ C_2\end{bmatrix}, \tag{12}$$

where C_1 and C_2 are constants.

§ 3. Numerical Solution and the Method of Coupling

By using the method of series expansion[6], we can numerically solve (10), the variational form of the system of highly singular integral equations. Divide the circumference into N and take the piecewise linear basis functions $\{L_j(\theta)\}\subset H^{\frac{1}{2}}(\Gamma)$, where $L_j(\theta_i)=\delta_{ij}$, $\theta_i=\frac{i}{N}2\pi$ $(i, j=1, \cdots, N)$. Let S_N be the subspace expanded by $\{L_j(\theta)\}$. We consider the approximate variational problem of (10):

$$\begin{cases} \text{Find } u_0^h\in S_N^2 \quad \text{such that} \\ \hat{D}(u_0^h, v_0^h)=\hat{F}(v_0^h), \quad \forall v_0^h\in S_N^2. \end{cases} \tag{13}$$

From $S_N^2\subset H^{\frac{1}{2}}(\Gamma)^2$ we know that the variational problem (13) has one and only one solution in $S_N^2/\mathscr{R}(\Gamma)$.

Let $u_0^h=\left[\sum_{j=1}^N U_jL_j(\theta),\ \sum_{j=1}^N V_jL_j(\theta)\right]^T$, where U_j, V_j $(j=1, 2, \cdots, N)$ are undetermined coefficients. Then from (13) we can obtain a system of linear algebraic equations

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix} U \\ V\end{bmatrix}=\begin{bmatrix} B \\ C\end{bmatrix}, \tag{14}$$

where

$$\begin{cases} U=[U_1, \cdots, U_N]^T, V=[V_1, \cdots, V_N]^T, B=[b_1, \cdots, b_N]^T, C=[c_1, \cdots, c_N]^T, \\ Q_{lm}=[q_{ij}^{(lm)}]_{i,j=1,\cdots,N}, \quad l, m=1, 2, \\ b_i=R\int_0^{2\pi} g_1(\theta)L_i(\theta)d\theta, \quad c_i=R\int_0^{2\pi} g_2(\theta)L_i(\theta)d\theta, \quad i=1, \cdots, N, \\ q_{ij}^{(11)}=\hat{D}(L_j, 0; L_i, 0), \quad q_{ij}^{(12)}=\hat{D}(0, L_j; L_i, 0), \\ q_{ij}^{(21)}=\hat{D}(L_j, 0; 0, L_i), \quad q_{ij}^{(22)}=\hat{D}(0, L_j; 0, L_i), \quad i, j=1, \cdots, N. \end{cases} \tag{15}$$

By direct computation we get

$$\begin{cases} Q_{11}=Q_{22}=\dfrac{2ab}{a+b}(a_0, a_1, \cdots, a_{N-1}), \\ Q_{12}=-Q_{21}=\dfrac{b^2}{a+b}(0, 1, 0, \cdots, 0, -1), \end{cases} \tag{16}$$

where $(\alpha_1, \cdots, \alpha_N)$ denotes the circulant matrix produced by $\alpha_1, \cdots, \alpha_N$,

$$a_k=\frac{4N^2}{\pi^5}\sum_{j=1}^{\infty}\frac{1}{j^5}\sin^4\frac{j\pi}{N}\cos\frac{jk}{N}2\pi, \quad k=0, 1, \cdots, N-1. \tag{17}$$

It is easily seen that Q_{11} and Q_{22} are symmetric circulant matrices, Q_{12} and Q_{21} are antisymmetric circulant matrices, and $Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is a semi-positive definite symmetric matrix.

From the existence and uniqueness of the solution of (13) in $S_N^2/\mathscr{R}(\Gamma)$, we know that the system (14) has a unique solution up to a vector in the space

$$\left\{\begin{bmatrix} U \\ V \end{bmatrix}\Big|_{U_1=U_2=\cdots=U_N,\ V_1=V_2=\cdots=V_N}\right\}.$$

Let $u_0\in H^{\frac{1}{2}}(\Gamma)$ be the solution of (8), and let $u_0^h\in S_N^2$ be its approximate solution obtained by the above numerical method. We have the following results.

Theorem 2. *If the interpolation operator Π: $H^{\frac{1}{2}}(\Gamma)\to S_N$ satisfies*

$$\lim_{N\to\infty}\|v_0-\Pi v_0\|_{\frac{1}{2},\Gamma}=0, \quad \forall v_0\in \text{ a dense subset of } H^{\frac{1}{2}}(\Gamma),$$

then

$$\lim_{N\to\infty}\|u_0-u_0^h\|_{\hat{D}}=0.$$

Theorem 3. *If the interpolation operator Π satisfies*

$$\|v-\Pi v\|_{s,\Gamma}\leqslant Ch^{k+1-s}|v|_{k+1,\Gamma}, \quad \forall v\in H^{k+1}(\Gamma), \ 0\leqslant s<k+1, \ k\geqslant 1,$$

and $u_0\in H^{k+1}(\Gamma)^2$, then

$$\|u_0-u_0^h\|_{\hat{D}}\leqslant Ch^{k+\frac{1}{2}}|u_0|_{k+1,\Gamma}; \tag{18}$$

if u_0 also satisfies $\langle u_0-u_0^h, v_0\rangle=0$, $\forall v_0\in\mathscr{R}(\Gamma)$, then

$$\|u_0-u_0^h\|_{L^2(\Gamma)^2}\leqslant Ch^{k+1}|u_0|_{k+1,\Gamma}, \tag{19}$$

where $h=\dfrac{2\pi}{N}$.

Estimates (18) and (19) are the best. In addition, using the inverse inequality in S_N, we can obtain another estimate, which is not the best,

$$\|u_0-u_0^h\|_{L^\infty(\Gamma)^2}\leqslant Ch^{k+\frac{1}{2}}|u_0|_{k+1,\Gamma}. \tag{20}$$

Particularly, when Π is the piecewise linear interpolation operator, we have (18)—(20) in which $k=1$.

The proof of Theorems 2 and 3 is classical[4,6,7], therefore it is omitted.

In combination with the classical finite element method, the system of canonical integral equations and the above numerical method can be used for solving general plane elasticity exterior boundary value problems. It is a method of coupling[3]. Let Ω, Γ and Γ' be the same as in Section 1. Γ' divides Ω into a bounded domain Ω_1 and an exterior circular domain Ω_2. Then the original problem (1) is equivalent to the following variational problem

$$\begin{cases} \text{Find } \boldsymbol{u}\in H^1(\Omega_1)^2 \quad \text{such that} \\ D_1(\boldsymbol{u}, \boldsymbol{v})+\hat{D}_2(\gamma'\boldsymbol{u}, \gamma'\boldsymbol{v})=F(\boldsymbol{v}), \quad \forall \boldsymbol{v}\in H^1(\Omega_1)^2, \end{cases} \tag{21}$$

where

$$D_1(\boldsymbol{u}, \boldsymbol{v})=\iint_{\Omega_1}\sum_{i,j=1}^{2}\sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\,dp,$$

$$\hat{D}_2(\boldsymbol{u}_0, \boldsymbol{v}_0)=\int_{\Gamma'}\boldsymbol{v}_0\cdot\mathscr{K}\boldsymbol{u}_0\,ds,$$

$$F(\boldsymbol{v})=\int_{\Gamma}\boldsymbol{g}\cdot\boldsymbol{v}\,ds,$$

ε_{ij} $(i, j=1, 2)$ are components of strain, γ' is the trace operator which maps $H^1(\Omega_1)^2$ onto $H^{\frac{1}{2}}(\Gamma')^2$, and $\mathscr{K}$ is the canonical integral operator defined by (8).

Now we divide circle Γ' into N_1 and take a triangulation of Ω_1 such that its nodes on Γ' coincide with the dividing points of Γ'. Let $S_h(\Omega_1)$ be the corresponding finite element functional space. From the characteristics of bilinear forms $D_1(\boldsymbol{u}, \boldsymbol{v})$ and $\hat{D}_2(\boldsymbol{u}_0, \boldsymbol{v}_0)$ we immediately obtain that the variational problem (21) and its approximate problem have unique solutions respectively in $H^1(\Omega_1)^2/\mathscr{R}(\Omega_1)$ and $S_h(\Omega_1)^2/\mathscr{R}(\Omega_1)$, where

$$\mathscr{R}(\Omega_1)=\{[c_1, c_2]\in H^1(\Omega_1)^2 \mid c_1 \text{ and } c_2 \text{ are constants}\}.$$

From the approximate form of (21) we can obtain a system of linear algebraic equations

$$(Q_1+Q_2)U=b, \tag{22}$$

where Q_1 is a general finite element stiffness matrix, and the 4 non-vanishing submatrices of Q_2 are given by (16)—(17); of course we must substitute N_1 for N.

By using the same method as in [3], we can obtain the convergence and error estimates for the method of coupling.

§ 4. The Approximation of Integral Boundary Condition

The system of canonical integral equations (8) is also a boundary condition on Γ'. In the same way as in [5], where a series of approximate forms of (5) has been given, the integral boundary condition (8) also can be approximated by a series of nonsingular integral boundary conditions.

Taking note of formulae (4), we naturally think of defining

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix} = \frac{1}{R\pi} \begin{bmatrix} \frac{2ab}{a+b}\sum_{n=1}^{M} n\cos n\theta & -\frac{2b^2}{a+b}\sum_{n=1}^{M} n\sin n\theta \\ \frac{2b^2}{a+b}\sum_{n=1}^{M} n\sin n\theta & \frac{2ab}{a+b}\sum_{n=1}^{M} n\cos n\theta \end{bmatrix} * \begin{bmatrix} u_1(R,\theta) \\ u_2(R,\theta) \end{bmatrix}$$

$$\equiv \mathscr{K}_M \begin{bmatrix} u_1(R,\theta) \\ u_2(R,\theta) \end{bmatrix} \tag{23}$$

as an approximate integral boundary condition of grade M of (8). Let

$$\hat{D}_2^M(\boldsymbol{u}_0, \boldsymbol{v}_0) = \int_{\Gamma'} \boldsymbol{v}_0 \cdot \mathscr{K}_M \boldsymbol{u}_0 \, ds.$$

We can easily get

Lemma 2. *$\hat{D}_2^M(\boldsymbol{u}_0, \boldsymbol{v}_0)$ is a semi-positive definite symmetric continuous bilinear form on $H^{\frac{1}{2}}(\Gamma')^2$.*

Its proof is much the same as the first part of the proof of Lemma 1.

Remark. It is more convenient for calculation to take

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix} = \frac{1}{R} \begin{bmatrix} \frac{2ab}{\pi(a+b)}\sum_{n=1}^{M} n\cos n\theta & \frac{2b^2}{a+b}\delta'(\theta) \\ -\frac{2b^2}{a+b}\delta'(\theta) & \frac{2ab}{\pi(a+b)}\sum_{n=1}^{M} n\cos n\theta \end{bmatrix} * \begin{bmatrix} u_1(R,\theta) \\ u_2(R,\theta) \end{bmatrix} \tag{24}$$

as an approximation of (8), but unfortunately, the bilinear form obtained from (24) is not semi-positive definite.

From Lemma 2 and the characteristics of bilinear $D_1(\boldsymbol{u}, \boldsymbol{v})$ we obtain

Theorem 4. *The variational problem*

$$\begin{cases} \text{Find } \boldsymbol{u}^M \in H^1(\Omega_1)^2 \quad \text{such that} \\ D_1(\boldsymbol{u}^M, \boldsymbol{v}) + \hat{D}_2^M(\gamma'\boldsymbol{u}^M, \gamma'\boldsymbol{v}) = F(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in H^1(\Omega_1)^2 \end{cases} \tag{25}$$

and its approximation

$$\begin{cases} \text{Find } \boldsymbol{u}_M^h \in S_h(\Omega_1)^2 \quad \text{such that} \\ D_1(\boldsymbol{u}_M^h, \boldsymbol{v}) + \hat{D}_2^M(\gamma'\boldsymbol{u}_M^h, \gamma'\boldsymbol{v}) = F(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in S_h(\Omega_1)^2 \end{cases} \tag{26}$$

have unique solutions respectively in $H^1(\Omega_1)^2/\mathscr{R}(\Omega_1)$ and $S_h(\Omega_1)^2/\mathscr{R}(\Omega_1)$.

From (26) we can also obtain a system of linear algebraic equations

$$(Q_1 + Q_2^M)U = b,$$

where Q_2^M contains only 4 non-vanishing submatrices as follows:

$$\begin{cases} Q_{11} = Q_{22} = \frac{2ab}{a+b}(a_0, a_1, \cdots, a_{N_1-1}), \\ Q_{12} = -Q_{21} = \frac{2b^2}{a+b}(0, d_1, \cdots, d_{N_1-1}), \end{cases} \tag{27}$$

$$\begin{cases} a_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{M}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\cos\frac{jk}{N}2\pi, \\ d_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{M}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\sin\frac{jk}{N}2\pi, \quad k=0, 1, \cdots, N_1-1. \end{cases} \tag{28}$$

Now we give the error estimate of the approximate solution $\boldsymbol{u}_M^h$.

Theorem 5. *If $\boldsymbol{u} \in H^k(\Omega_1)^2$, $\gamma'\boldsymbol{u} \in H^{k-\frac{1}{2}}(\Gamma')^2$, then*

$$\|u-u_M^h\|_{1,\Omega_1}\leqslant C\Big(h\|u\|_{2,\Omega_1}+\frac{1}{M^{k-1}}\|u\|_{k-\frac{1}{2},\Gamma'}\Big), \tag{29}$$

where C is a constant independent of h and M.

Proof. According to Theorem 2.4 of [5], we have

$$\|u-u_M^h\|_{1,\Omega_1}\leqslant C\Big\{\inf_{v\in S_h^1}\|u-v\|_{1,\Omega_1}+\sup_{w\in S_h^1}\frac{|\hat{D}_2^M(\gamma'u,\gamma'w)-\hat{D}_2(\gamma'u,\gamma'w)|}{\|w\|_{1,\Omega_1}}\Big\}.$$

For the first term, we have a classical result[10]

$$\inf_{v\in S_h^1}\|u-v\|_{1,\Omega_1}\leqslant Ch\|u\|_{2,\Omega_1}.$$

For the second term, let $u=[u_1,u_2]^T$, $w=[w_1,w_2]^T$,

$$u_1(R,\theta)=\sum_{-\infty}^{\infty}a_ne^{in\theta},\ a_{-n}=\bar{a}_n,\ u_2(R,\theta)=\sum_{-\infty}^{\infty}b_ne^{in\theta},\ b_{-n}=\bar{b}_n,$$

$$w_1(R,\theta)=\sum_{-\infty}^{\infty}c_ne^{in\theta},\ c_{-n}=\bar{c}_n,\ w_2(R,\theta)=\sum_{-\infty}^{\infty}d_ne^{in\theta},\ d_{-n}=\bar{d}_n.$$

We have

$$\begin{aligned}
&|\hat{D}_2^M(\gamma'u,\gamma'w)-\hat{D}_2(\gamma'u,\gamma'w)|\\
&=2\pi\sum_{n=M+1}^{\infty}[\bar{c}_n,\bar{d}_n]\begin{bmatrix}\dfrac{2ab}{a+b}|n| & \dfrac{2b^2}{a+b}in\\ -\dfrac{2b^2}{a+b}in & \dfrac{2ab}{a+b}|n|\end{bmatrix}\begin{bmatrix}a_n\\ b_n\end{bmatrix}\\
&\leqslant 2\pi\sum_{n=M+1}^{\infty}|n|\frac{2b}{a+b}[a(|a_n||c_n|+|b_n||d_n|)+b(|b_n||c_n|+|a_n||d_n|)]\\
&\leqslant 2\pi\sum_{n=M+1}^{\infty}|n|2b(|a_n|^2+|b_n|^2)^{\frac{1}{2}}(|c_n|^2+|d_n|^2)^{\frac{1}{2}}\\
&\leqslant\frac{4b\pi}{M^{k-1}}\sum_{n=M+1}^{\infty}|n|^k(|a_n|^2+|b_n|^2)^{\frac{1}{2}}(|c_n|^2+|d_n|^2)^{\frac{1}{2}}\\
&\leqslant\frac{4b\pi}{M^{k-1}}\sum_{n=M+1}^{\infty}[(1+n^2)^{k-\frac{1}{2}}(|a_n|^2+|b_n|^2)]^{\frac{1}{2}}[(1+n^2)^{\frac{1}{2}}(|c_n|^2+|d_n|^2)]^{\frac{1}{2}}\\
&\leqslant\frac{C}{M^{k-1}}\|u\|_{k-\frac{1}{2},\Gamma'}\|w\|_{\frac{1}{2},\Gamma'}\leqslant\frac{C}{M^{k-1}}\|u\|_{k-\frac{1}{2},\Gamma'}\|w\|_{1,\Omega_1}.
\end{aligned}$$

Then inequality (29) follows immediately.

From the proof of Theorem 5 we see that, in fact, $\|u\|_{k-\frac{1}{2},\Gamma'}$ in (29) can be replaced by $\|R_Mu\|_{k-\frac{1}{2},\Gamma'}$, where

$$R_Mu=[\sum_{|n|\geqslant M+1}a_ne^{in\theta},\quad\sum_{|n|\geqslant M+1}b_ne^{in\theta}]^T.$$

Therefore the error has a relation to the characteristic of the Fourier expansion of $\gamma'u$.

§ 5. Numerical Example

Let Ω be the exterior domain to the unit circle. We want to solve the system of canonical integral equations (8), where $a=2$, $b=0.5$,

$$g_1=0.6\sin\theta,\quad g_2=0.6\cos\theta.$$

The exact solution of the corresponding plane elasticity problem is

$$\begin{cases} u_1(r,\theta)=r^{-1}(1.6\sin 2\theta\cos\theta-0.4\cos 2\theta\sin\theta)-0.6r^{-3}\sin 3\theta, \\ u_2(r,\theta)=r^{-1}(1.6\sin 2\theta\sin\theta+0.4\cos 2\theta\cos\theta)+0.6r^{-3}\cos 3\theta. \end{cases}$$

Applying the numerical method given in Section 3 and the approximate formulae (27)—(28) of grade M, we get the following results, where N is the unmber of nodes on Γ.

M	Error \ N	4	8	16	32
2	$\|u-u^h_k\|_{L_\infty(\Gamma)}$	0.2337008	0.0530297	0.0129502	0.0032231
	Ratio	4.406979	4.094894	4.017933	
	$\|u-u^h_k\|_{L_2(\Gamma)}$	0.3036854	0.0619294	0.0146808	0.0036624
	Ratio	4.903736	4.218394	4.008519	
5	$\|u-u^h_k\|_{L_\infty(\Gamma)}$	0.1532836	0.0530289	0.0129508	0.0032246
	Ratio	2.890567	4.094643	4.016250	
	$\|u-u^h_k\|_{L_2(\Gamma)}$	0.3275518	0.0619294	0.0146808	0.0036622
	Ratio	5.289116	4.218394	4.008738	

M	Error \ N	16	32
100	$\|u-u^h_k\|_{L_\infty(\Gamma)}$	0.0121152	0.0030972
	Ratio	3.911662	
	$\|u-u^h_k\|_{L_2(\Gamma)}$	0.0148136	0.0036744
	Ratio	4.031569	

As we know, the necessary grade M of approximation has a relation to the Fourier expansion of the exact solution; therefore it has a relation to that of the given boundary traction. From the above tables we can see that, for this example, it is already sufficient that $M=2$. Since almost all values of the ratio are about 4, we conclude that these approximate solutions have an accuracy of order about $O(h^2)$, whether in L_∞ norm or in L_2 norm.

Because Q_{ij} $(i, j=1, 2)$ are symmetric or anti-symmetric circulant matrices, and because the necessary grade M of approximation is very small in general, the calculation is quite economical.

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