THE UNSOLVABILITY OF INVERSE ALGEBRAIC EIGENVALUE PROBLEMS ALMOST EVERYWHERE*

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Abstract

This paper revises the definition for the unsolvability of inverse algebraic eigenvalue problems almost everywhere (a.e.) given by Shapiro [5], and gives some sufficient and necessary conditions such that the inverse algebraic eigenvalue problems are unsolvable a.e.

§ 1. Introduction

The general inverse algebraic eigenvalue problems are the following problems:

Problem G-1. Given $m+1$ real $n\times n$ symmetric matrices $A, A_1, \cdots, A_m$, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n (k\geqslant 1)$, find a real m-dimensional vector $c=(c_1, \cdots, c_m)^T$ such that the matrix $A+\sum\limits_{i=1}^{m} c_iA_i$ has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (see [2], [5], [7]).

Problem A-1. Given a real $n\times n$ symmetric matrix A, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real n-dimensional vector $c=(c_1, \cdots, c_n)^T$ such that the matrix $A+\operatorname{diag}(c_1, \cdots, c_n)$ has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (see [2], [3]).

Problem G-2. Given $m+1$ real $n\times n$ matrices $A_0, A_1, \cdots, A_m$, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n (k\geqslant 1)$, find a real m-dimensional vector $c=(c_1, \cdots, c_m)^T$ such that the matrix $A+\sum\limits_{i=1}^{m} c_iA_i$ is diagonalizable and has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (ref. [2]).

Problem A-2. Given a real $n\times n$ matrix A, k non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real n-dimensional vector $c=(c_1, \cdots, c_n)^T$ such that the matrix $A+\operatorname{diag}(c_1, \cdots, c_n)$ is diagonalizable and has zero eigenvalue of multiplicity r_0 and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively (ref. [2], [3]).

Problems A-1 and A-2 are additive inverse eigenvalue problems. Problems G-1 and G-2 are general inverse eigenvalue problems. Some of these problems arise often

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in applied mathematics, and are studied by many authors (see [2]—[5], [7] and the references contained therein).

Recently A. Shapiro [5] has defined the unsolvability of Problem G–1 almost everywhere (a.e.), and has given a sufficient condition such that Problem G–1 is unsolvable a.e. By [5] Problem G–1 is said to be unsolvable a.e. if the set of matrices A (in the vector space of $n\times n$ symmetric matrices) at which it is solvable has measure zero. Shapiro [5] has proved the following conclusion.

Theorem 1.1. Problem G–1 is unsolvable a.e., if

$$\sum_{t=0}^{k}\frac{r_t(r_t+1)}{2}>m+k. \tag{1.1}$$

Undoubtedly, the study of the unsolvability of inverse eigenvalue problems a.e. is important. But it seems that the above mentioned definition given by Shapiro [5] is not enough to clarify the concept of unsolvability a.e. and condition (1.1) looks too strong. In this paper we give a more reasonable definition for the unsolvability of inverse eigenvalue problems a.e., and give some sufficient and necessary conditions such that Problems G–1, A–1, G–2 and A–2 are unsolvable a.e. respectively.

This paper uses the following notation. The symbol $\mathbb{R}^{m\times n}$ denotes the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. $I^{(n)}$ is the $n\times n$ identity matrix, and O is the null matrix. The superscript T is for transpose, and

$$S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^T=A\},\quad \mathbb{O}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^TA=I\},$$
$$S\mathbb{R}_0^{n\times n}=\{A=(a_{ij})\in S\mathbb{R}^{n\times n}:\ a_{ii}=0,\ 1\leqslant i\leqslant n\},$$
$$\mathbb{R}_0^{n\times n}=\{A=(a_{ij})\in\mathbb{R}^{n\times n}:\ a_{ii}=0,\ 1\leqslant i\leqslant n\}.$$

Besides, for $A=(a_{ij})\in\mathbb{R}^{n\times n}$ we write

$$|A|=(|a_{ij}|),\quad k_1(A)=\max_{1<j<n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|\Big)$$

and

$$k_2(A)=\max_{1<j<n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}a_{ij}^2\Big)^{1/2}.$$

Now we define the unsolvability of inverse eigenvalue problems a.e.

Definition 1.1. *Problem G–1 is said to be unsolvable almost everywhere (u.s.a.e.) if the set of matrices $A, A_1, \cdots, A_m\in S\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{m+1}\times\mathbb{R}^k$.*

Definition 1.2. *Problem A–1 is said to be u.s.a.e. if the set of matrices $A\in S\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $S\mathbb{R}^{n\times n}\times\mathbb{R}^k$.*

Definition 1.3. *Problem G–2 is said to be u.s.a.e. if the set of matrices $A, A_1, \cdots, A_m\in\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m+1}\times\mathbb{R}^k$.*

Definition 1.4. *Problem A–2 is said to be u.s.a.e. if the set of matrices $A\in\mathbb{R}^{n\times n}$ and vectors $\lambda\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $\mathbb{R}^{n\times n}\times\mathbb{R}^k$.*

In passing we explain that a set $\mathscr{L}$ of $\mathbf{R}^N$ is said to be of measure zero (meas$\mathscr{L}=0$) if for any $\varepsilon>0$, there exists a sequence of open set $\mathscr{B}_i^\varepsilon$ such that

$$\mathscr{L}\subset\bigcup_{i=1}^{\infty}\mathscr{B}_i^\varepsilon, \quad \sum_{i=1}^{\infty}\mathrm{Volume}(\mathscr{B}_i^\varepsilon)<\varepsilon$$

(ref. [6], 45—46).

§ 2. Main Results

Theorem 2.1. *Problem G–1 is u.s.a.e. if*

$$n-m+\frac{r(r-1)}{2}>0, \tag{2.1}$$

where $r=\max\{r_0, r_1, \cdots, r_k\}$. In addition, if $m=n$ then $r>1$ is a sufficient and necessary condition for the unsolvability of Problem G–1 a.e.

Theorem 2.2. *Problem A–1 is u.s.a.e. if and only if*

$$\max\{r_0, r_1, \cdots, r_k\}>1. \tag{2.2}$$

Theorem 2.3. *Problem G–2 is u.s.a.e. if*

$$n-m+r(r-1)>0, \tag{2.3}$$

where $r=\max\{r_0, r_1, \cdots, r_k\}$. In addition, if $m=n$ then $r>1$ is a sufficient and necessary condition for the unsolvability of Problem G–2 a.e.

Theorem 2.4. *Problem A–2 is u.s.a.e. if and only if*

$$\max\{r_0, r_1, \cdots, r_k\}>1. \tag{2.4}$$

§ 3. Proofs of Theorem 2.1—Theorem 2.4

The proofs of Theorem 2.1—Theorem 2.4 will be based on a submanifold theorem (ref. [1], 39—42, Definition II. 2.1 and Theorem II. 2.1) and on Sard's theorem (ref. [6], 45—55, [5]) which we shall cite here.

Theorem 3.1[1]. *Let $f=(f_1, \cdots, f_r)$ be a differentiable vector-value function defined in $\mathbf{R}^n$, and let $\mathfrak{M}\subset\mathbf{R}^n$ be the set of points $x=(\xi_1, \cdots, \xi_n)^T\in\mathbf{R}^n$ such that*

$$f_1(x)=0, \cdots, f_r(x)=0.$$

Assume that for each point $x\in\mathfrak{M}$ the matrix

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial f_1}{\partial \xi_1} & \cdots & \frac{\partial f_r}{\partial \xi_1}\\ \vdots & & \vdots\\ \frac{\partial f_1}{\partial \xi_n} & \cdots & \frac{\partial f_r}{\partial \xi_n}\end{pmatrix},$$

where each partial derivative is evaluated at x has rank r. Then $\mathfrak{M}$ is an $n-r$-dimensional submanifold of $\mathbf{R}^n$.

Definition 3.1[5],[6]. *Let $\mathfrak{M}$ and $\mathscr{K}$ be differentiable manifolds, and let F be a differentiable mapping of $\mathfrak{M}\to\mathscr{K}$. A point $x\in\mathfrak{M}$ is said to be regular if the differential dF of F at x (as a linear transformation from the tangent space $T_x(\mathfrak{M})$ to the tangent space $T_y(\mathscr{K})$, $y=F(x)$) is onto, i.e., its range is the whole space $T_y(\mathscr{K})$. The non-regular points of $\mathfrak{M}$ are called critical. A point $y\in\mathscr{K}$ such that $F^{-1}(y)$ contains at* *least one critical point is called a critical value.*

Theorem 3.2 (*Sard's theorem*)[5],[6]. *Let $\mathfrak{M}$ and $\mathscr{K}$ be differentiable manifolds, and let F be an infinitely differentiable mapping of $\mathfrak{M}\to\mathscr{K}$. Then the critical values of F form a set of measure zero.*

Utilizing Theorem 3.2 we can prove the following lemma.

Lemma 3.1. *Let $\mathfrak{M}$ be an m-dimensional differentiable submanifold of an n-dimensional Euclidean space $\mathscr{E}$, $\mathscr{K}$ be a k-dimensional Euclidean space, $k<n$, and let F be a differentiable mapping of $\mathfrak{M}\to\mathscr{K}$ defined by the following expressions:*

$$\eta_1=\xi_1,\ \cdots,\ \eta_k=\xi_k,\quad \text{for}\quad x=(\xi_1,\ \cdots,\ \xi_n)^T\in\mathfrak{M}\quad \text{and}\quad y=(\eta_1,\ \cdots,\ \eta_k)^T\in\mathscr{K}.$$

Then $F(\mathfrak{M})$ is a set of measure zero in $\mathscr{K}$ if $m<k$.

Proof. By Theorem 3.2 it will be enough to show that all values of F are critical.

Let $x^{(0)}=(\xi_1^{(0)},\ \cdots,\ \xi_n^{(0)})^T$ be any point of $\mathfrak{M}$. Then there exist a neighbourhood $\mathscr{U}(x^{(0)})$ of $x^{(0)}$ and a differentiable mapping $\varphi=(\varphi_1,\ \cdots,\ \varphi_n)$ of an open ball $\mathscr{B}$ of m-dimensional Euclidean space $\mathscr{E}_m$ onto $\mathscr{U}(x^{(0)})\cap\mathfrak{M}$. According to the hypotheses we have

$$F:\ \eta_i=\varphi_i(\tau_1,\cdots,\ \tau_m),\ i=1,\ \cdots,\ k,\ t=(\tau_1,\cdots,\ \tau_m)^T\in\mathscr{B},\ y=(\eta_1,\ \cdots,\ \eta_k)^T\in\mathscr{K},$$

and

$$\eta_i^{(0)}=\varphi_i(\tau_1^{(0)},\ \cdots,\ \tau_m^{(0)}),\ i=1,\ \cdots,\ k,\ t^{(0)}=(\tau_1^{(0)},\ \cdots,\ \tau_m^{(0)})^T\in\mathscr{B},\ y^{(0)}=(\eta_1^{(0)},\ \cdots,\ \eta_k^{(0)})^T\in\mathscr{K}.$$

And thus the differential dF of F at $x^{(0)}$ can be expressed by

$$d\eta_1=\sum_{i=1}^{m}\left(\frac{\partial\varphi_1}{\partial\tau_i}\right)_{t=t^{(0)}}d\tau_i,\ \cdots,\ d\eta_k=\sum_{i=1}^{m}\left(\frac{\partial\varphi_k}{\partial\tau_i}\right)_{t=t^{(0)}}d\tau_i.$$

Therefore the necessary condition for the differential dF of F at $x^{(0)}$ to be onto is that the corresponding linear equations in $d\tau_1,\ \cdots,\ d\tau_m$,

$$\begin{pmatrix}\left(\frac{\partial\varphi_1}{\partial\tau_1}\right)_{t=t^{(0)}} & \cdots & \left(\frac{\partial\varphi_1}{\partial\tau_m}\right)_{t=t^{(0)}}\\ \vdots & & \vdots\\ \left(\frac{\partial\varphi_k}{\partial\tau_1}\right)_{t=t^{(0)}} & \cdots & \left(\frac{\partial\varphi_k}{\partial\tau_m}\right)_{t=t^{(0)}}\end{pmatrix}\begin{pmatrix}d\tau_1\\ \vdots\\ d\tau_m\end{pmatrix}=\begin{pmatrix}d\eta_1\\ \vdots\\ d\eta_k\end{pmatrix}, \tag{3.1}$$

are solvable for any differential elements $d\eta_1,\ \cdots,\ d\eta_k$. But observe that the system of equations (3.1) has k linear equations with m unknowns. Therefore in the case of $m<k$ there are differential elements $d\eta_1,\ \cdots,\ d\eta_k$ such that the system of equations (3.1) is unsolvable and hence any value of F is critical. ∎

Now we prove Theorem 2.1—Theorem 2.4.

Proof of Theorem 2.1.

1) First observe that if Problem G–1 is solvable at $A,\ A_1,\ \cdots,\ A_m\in S\mathbb{R}^{n\times n}$ and $\lambda\in\mathbb{R}^k$, then there exist a matrix $U\in\mathbb{O}^{n\times n}$ and a vector $c=(c_1,\ \cdots,\ c_m)^T\in\mathbb{R}^m$ such that

$$A+\sum_{i=1}^{m}c_iA_i=U\ \mathrm{diag}(O^{(r_0)},\ \lambda_1I^{(r_1)},\ \cdots,\ \lambda_kI^{(r_k)})U^T, \tag{3.2}$$

where

$$A=(a_{ij}),\ A_t=(a_{ij}^{(t)}),\ t=1,\ \cdots,\ m,\ U=(u_1,\ \cdots,\ u_n). \tag{3.3}$$

Suppose that $r_j=\max\{r_0,\ r_1,\ \cdots,\ r_k\}$ for some index $j\in\{0,\ 1,\ \cdots,\ k\}$. Then

$$(A-\lambda_j I)+\sum_{t=1}^{m} c_t A_t = U\,\mathrm{diag}(-\lambda_j I^{(r_0)}, (\lambda_1-\lambda_j)I^{(r_1)}, \cdots, O^{(r_j)}, \cdots, (\lambda_k-\lambda_j)I^{(r_k)})U^T.$$

Therefore we may assume without loss of generality that

$$r_0=\max\{r_0, r_1, \cdots, r_k\}.$$

Let

$$a=(a_{11}, a_{12}, \cdots, a_{1n}, a_{22}, a_{23}, \cdots, a_{2n}, \cdots, a_{nn})^T\in \mathbb{R}^{\frac{n(n+1)}{2}}, \tag{3.4}$$

$$a_t=(a_{11}^{(t)}, a_{12}^{(t)}, \cdots, a_{1n}^{(t)}, a_{22}^{(t)}, a_{23}^{(t)}, \cdots, a_{2n}^{(t)}, \cdots, a_{nn}^{(t)})^T\in \mathbb{R}^{\frac{n(n+1)}{2}}, \quad 1\leqslant t\leqslant m, \tag{3.5}$$

$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in \mathbb{R}^k, \quad c=(c_1, \cdots, c_m)^T\in \mathbb{R}^m, \tag{3.6}$$

$$u=(u_{r_0+1}^T, \cdots, u_n^T)^T\in \mathbb{R}^{n(n-r_0)} \tag{3.7}$$

and

$$\mathscr{E}=\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{m+1}\times \mathbb{R}^k\times \mathbb{R}^m\times \mathbb{R}^{n(n-r_0)}.$$

We define differentiable real-valued functions $g_{ij}(1\leqslant i, j\leqslant n)$ and $h_{ij}(1\leqslant i, j\leqslant n-r_0)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{ij})=A+\sum_{t=1}^{m} c_t A_t-(u_{r_0+1}, \cdots, u_n)\mathrm{diag}(\lambda_1 I^{(r_1)}, \cdots, \lambda_k I^{(r_k)})(u_{r_0+1}, \cdots, u_n)^T,$$

$$(h_{ij})=(u_{r_0+1}, \cdots, u_n)^T(u_{r_0+1}, \cdots, u_n)-I^{(n-r_0)}.$$

Set

$$g=(g_{11}, g_{12}, \cdots, g_{1n}, g_{22}, g_{23}, \cdots, g_{2n}, \cdots, g_{nn}),$$

$$h=(h_{11}, h_{12}, \cdots, h_{1,n-r_0}, h_{22}, h_{23}, \cdots, h_{2,n-r_0}, \cdots, h_{n-r_0,n-r_0})$$

and

$$f=(g, h).$$

Let $\mathfrak{M}\subset\mathscr{E}$ be the set of points $X=\{A, A_1, \cdots, A_m, \lambda, c, u\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A, A_1, \cdots, A_m, \lambda, c, u\}\in\mathscr{E}$ is associated a vector

$$x=(a^T, a_1^T, \cdots, a_m^T, \lambda^T, c^T, u^T)^T\in \mathbb{R}^{\frac{n(n+1)(m+1)}{2}+k+m+n(n-r_0)},$$

where $A, A_1, \cdots, A_m, a, a_1, \cdots, a_m, \lambda, c, u$ are represented by (3.3)—(3.7). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix} \frac{\partial g}{\partial a} & \frac{\partial h}{\partial a} \\ \frac{\partial g}{\partial \tilde{a}} & \frac{\partial h}{\partial \tilde{a}} \\ \frac{\partial g}{\partial \lambda} & \frac{\partial h}{\partial \lambda} \\ \frac{\partial g}{\partial c} & \frac{\partial h}{\partial c} \\ \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u} \end{pmatrix}=\begin{pmatrix} I^{(\frac{n(n+1)}{2})} & 0 \\ * & 0 \\ * & 0 \\ * & 0 \\ * & H \end{pmatrix},$$

where

$$\tilde{a}=(a_1^T, \cdots, a_m^T)^T\in \mathbb{R}^{\frac{mn(n+1)}{2}}$$

and

$$H=\frac{\partial h}{\partial u}=$$

$$\begin{pmatrix}
2u_{r_0+1} & u_{r_0+2} & u_{r_0+3} & u_{r_0+4} & \cdots & u_{n-2} & u_{n-1} & u_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & u_{r_0+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 2u_{r_0+2} & u_{r_0+3} & u_{r_0+4} & \cdots & u_{n-2} & u_{n-1} & u_n & \cdots & 0 & 0 & 0 \\
0 & 0 & u_{r_0+1} & 0 & \cdots & 0 & 0 & 0 & 0 & u_{r_0+2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & u_{r_0+1} & 0 & 0 & 0 & 0 & 0 & \cdots & u_{r_0+2} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & u_{r_0+1} & 0 & 0 & 0 & 0 & \cdots & 0 & u_{r_0+2} & 0 & \cdots & 2u_{n-1} & u_n & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_{r_0+1} & 0 & 0 & 0 & \cdots & 0 & 0 & u_{r_0+2} & \cdots & 0 & u_{n-1} & 2u_n
\end{pmatrix}. \tag{3.8}$$

From

$$H^T H=\operatorname{diag}(4,\ 2I^{(n-r_0-1)},\ 4,\ 2I^{(n-r_0-2)},\ \cdots,\ 4,\ 2,\ 4)$$

we see that

$$\operatorname{rank}(H)=\frac{(n-r_0)(n-r_0+1)}{2}.$$

Hence for each point $X\in\mathfrak{M}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at X has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=\frac{n(n+1)}{2}+\frac{(n-r_0)(n-r_0+1)}{2}.$$

By Theorem 3.1, it follows from

$$\dim(\mathscr{E})=\frac{n(n+1)(m+1)}{2}+k+m+n(n-r_0)$$

that $\mathfrak{M}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=\frac{mn(n+1)}{2}+k+m+\frac{(n-r_0)(n+r_0-1)}{2}. \tag{3.9}$$

Let

$$\mathscr{X}=\underbrace{S\mathsf{R}^{n\times n}\times\cdots\times S\mathsf{R}^{n\times n}}_{m+1}\times\mathsf{R}^k,$$

and let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A,\ A_1,\ \cdots,\ A_m,\ \lambda\}\in\mathscr{X}$ at which Problem G-1 is solvable. We define a differentiable mapping F of $\mathfrak{M}\to\mathscr{X}$:

$$F(X)=\{A,\ A_1,\ \cdots,\ A_m,\ \lambda\}\in\mathscr{X}\quad\text{for}\quad X=\{A,\ A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ u\}\in\mathfrak{M},$$

and write

$$\mathfrak{M}'=F(\mathfrak{M}).$$

Since

$$\dim(\mathscr{X})=\frac{n(n+1)(m+1)}{2}+k,$$

and $\dim(\mathfrak{M})<\dim(\mathscr{X})$ if

$$n-m+\frac{r_0(r_0-1)}{2}>0, \tag{3.10}$$

by Lemma 3.1 the set $\mathfrak{M}'$ has measure zero under the assumption of (3.10). Observe that for any point $X^{(1)}=\{A, A_1, \cdots, A_m, \lambda\}\in\mathscr{L}$ there exist $c\in\mathbb{R}^m$ and $u\in\mathbb{R}^{n(n-r_0)}$ such that the point $X=\{A, A_1, \cdots, A_m, \lambda, c, u\}\in\mathfrak{M}$. Hence $\mathscr{L}\subset\mathfrak{M}'$, and thus the set $\mathscr{L}$ has measure zero in the space $\mathscr{K}$. This means that Problem G-1 is u.s.a.e. if the condition (2.1) is fulfilled.

2) We consider the case $m=n$. Let

$$\mathscr{K}=\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{n+1}\times\mathbb{R}^n,$$

and

$$\mathscr{K}_s=\{\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{K}:\ A_t=(a_{ij}^{(t)}),\ 1<a_{ii}^{(i)}<2,\ |a_{ii}^{(t)}|<s,\ i\neq t,\ 1\leqslant i,\ t\leqslant n\},$$

where s is a fixed positive number satisfying $s\ll\frac{1}{n-1}$. Obviously, $\mathscr{K}_s$ is an open set of the Euclidean space $\mathscr{K}$, and the matrix

$$E=(a_{ii}^{(j)})_{i,j=1,\cdots,n}$$

is nonsingular provided $\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{K}_s$. Let

$$E^{-1}=(l'_{ij}),\qquad \mu=\sup_{\substack{\{A, A_1,\cdots,A_n,\lambda\}\in\mathscr{K}_s\\ 1\leqslant i,j\leqslant n}}|l'_{ij}|,$$

$$\tilde{A}_t=\sum_{i=1}^n l'_{it}A_i,\quad t=1,\cdots,n$$

and

$$S=\sum_{t=1}^n|A_t|,\quad \tilde{S}=\sum_{t=1}^n|\tilde{A}_t|\quad \forall\{A, A_1,\cdots,A_n,\lambda\}\in\mathscr{K}_s,$$

$$d(\lambda)=\min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}|\lambda_i-\lambda_j|\quad \forall\lambda=(\lambda_1,\cdots,\lambda_n)^T\in\mathbb{R}^n,$$

$$a_0=(a_{11},\cdots,a_{nn})^T\in\mathbb{R}^n\quad \forall A=(a_{ij})\in S\mathbb{R}^{n\times n},$$

$$k^{(2)}=n\mu\|\lambda-a_0\|_\infty k_2(S)+k_2(A)\quad \forall\{A, A_1,\cdots,A_n,\lambda\}\in\mathscr{K}_s$$

and

$$\mathscr{K}^*=\{\{A, A_1,\cdots,A_n,\lambda\}\in\mathscr{K}_s:\ d(\lambda)>2k^{(2)}\{[3+n^2\mu^2k_2(S)^2]^{\frac{1}{2}}+n\mu k_2(S)\}\}.$$

From

$$k_2(\tilde{S})\leqslant n\mu k_2(S)\quad \forall\{A, A_1,\cdots,A_n,\lambda\}\in\mathscr{K}_s$$

we see that if $r=1$ and $\{A, A_1,\cdots,A_n,\lambda\}\in\mathscr{K}^*$ then by Theorem 6 of [2] Problem G-1 has a solution $c=(c_1,\cdots,c_n)^T\in\mathbb{R}^n$. Observe that $\mathscr{K}^*$ is a nonempty open set of $\mathscr{K}$ and so meas $\mathscr{K}^*>0$ if $r=1$. Hence in the case $m=n$ the inequality $r>1$ is not only a sufficient but also a necessary condition for the unsolvability of Problem G-1 a.e. ∎

Proof of Theorem **2.2.**

1) Suppose that Problem A-1 is solvable at $A\in S\mathbb{R}^{n\times n}$ and $\lambda\in\mathbb{R}^k$. Then there exist a matrix $U\in\mathbb{O}^{n\times n}$ and a vector $c=(c_1,\cdots,c_n)^T\in\mathbb{R}^n$ such that

$$A+\operatorname{diag}(c_1,\cdots,c_n)=U\operatorname{diag}(O^{(r_0)},\lambda_1I^{(r_1)},\cdots,\lambda_kI^{(r_k)})U^T,\tag{3.11}$$

where

$$A=(a_{ij}),\quad U=(u_1,\cdots,u_n).\tag{3.12}$$

Without loss of generality we may assume that $r_0=\max\{r_0, r_1, \cdots, r_k\}$ and $A\in SR_0^{n\times n}$.

Let

$$a=(a_{12}, a_{13}, \cdots, a_{1n}, a_{23}, a_{24}, \cdots, a_{2n}, \cdots, a_{n-1,n})^T\in \mathbb{R}^{\frac{n(n-1)}{2}}, \tag{3.13}$$

$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in \mathbb{R}^k, \tag{3.14}$$

$$u=(u_{r_0+1}^T, \cdots, u_n^T)^T\in \mathbb{R}^{n(n-r_0)} \tag{3.15}$$

and

$$\mathscr{E}=SR_0^{n\times n}\times \mathbb{R}^k\times \mathbb{R}^{n(n-r_0)}.$$

We define differentiable real-valued functions $g_{ij}(1\leqslant i, j\leqslant n)$ and $h_{ij}(1\leqslant i, j\leqslant n-r_0)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{ij})=A-(u_{r_0+1}, \cdots, u_n)\operatorname{diag}(\lambda_1 I^{(r_1)}, \cdots, \lambda_k I^{(r_k)})(u_{r_0+1}, \cdots, u_n)^T,$$

$$(h_{ij})=(u_{r_0+1}, \cdots, u_n)^T(u_{r_0+1}, \cdots, u_n)-I^{(n-r_0)}.$$

Set

$$g=(g_{12}, g_{13}, \cdots g_{1n}, g_{23}, g_{24}, \cdots, g_{2n}, \cdots, g_{n-1,n}),$$

$$h=(h_{11}, h_{12}, \cdots, h_{1,n-r_0}, h_{22}, h_{23}, \cdots, h_{2,n-r_0}, \cdots, h_{n-r_0,n-r_0})$$

and

$$f=(g, h).$$

Let $\mathfrak{M}\subset\mathscr{E}$ be the set of points $X=\{A, \lambda, u\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A, \lambda, u\}\in\mathscr{E}$ is associated a vector

$$x=(a^T, \lambda^T, u^T)^T\in \mathbb{R}^{\frac{n(n-1)}{2}+k+n(n-r_0)},$$

where A, a, λ, u are represented by (3.12)—(3.15). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\dfrac{\partial g}{\partial a} & \dfrac{\partial h}{\partial a}\\ \dfrac{\partial g}{\partial \lambda} & \dfrac{\partial h}{\partial \lambda}\\ \dfrac{\partial g}{\partial u} & \dfrac{\partial h}{\partial u}\end{pmatrix}=\begin{pmatrix}I^{\left(\frac{n(n-1)}{2}\right)} & 0\\ * & 0\\ * & H\end{pmatrix},$$

where H is the matrix denoted by (3.8) and $\operatorname{rank}(H)=(n-r_0)(n-r_0+1)/2$. Hence for each point $X\in\mathfrak{M}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at X has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=\frac{n(n-1)}{2}+\frac{(n-r_0)(n-r_0+1)}{2}.$$

By Theorem 3.1, it follows from

$$\dim(\mathscr{E})=\frac{n(n-1)}{2}+k+n(n-r_0)$$

that $\mathfrak{M}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=k+\frac{(n-r_0)(n+r_0-1)}{2}. \tag{3.16}$$

Let

$$\mathscr{K}=SR_0^{n\times n}\times \mathbb{R}^k,$$

and let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A, \lambda\}\in\mathscr{X}$ at which Problem A-1 is solvable. We define a differentiable mapping F of $\mathfrak{M}\to\mathscr{X}$:

$$F(X)=\{A, \lambda\}\in\mathscr{X} \quad \text{for} \quad X=\{A, \lambda, u\}\in\mathfrak{M},$$

and write

$$\mathfrak{M}'=F(\mathfrak{M}).$$

Since

$$\dim(\mathscr{X})=\frac{n(n-1)}{2}+k,$$

and $\dim(\mathfrak{M})<\dim(\mathscr{X})$ if

$$r_0>1, \tag{3.17}$$

by Lemma 3.1 the set $\mathfrak{M}'$ has measure zero under the assumption of (3.17). Observe that for any point $X^{(1)}=\{A, \lambda\}\in\mathscr{L}$ there exists $u\in\mathsf{R}^{n(n-r_0)}$ such that the point $X=\{A, \lambda, u\}\in\mathfrak{M}$. Hence $\mathscr{L}\subset\mathfrak{M}'$, and thus the set $\mathscr{L}$ has measure zero in the space $\mathscr{X}$. This means that Problem A-1 is u.s.a.e. if the condition (2.2) is fulfilled.

2) Let

$$\mathscr{X}=S\mathsf{R}_0^{n\times n}\times\mathsf{R}^n,$$

$$d(\lambda)=\min_{\substack{1\le i,j\le n\\ i\ne j}}|\lambda_i-\lambda_j| \quad \lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathsf{R}^n$$

and

$$\mathscr{X}^*=\{\{A, \lambda\}\in\mathscr{X}:\ d(\lambda)>2\sqrt{3}\ k_2(A)\}.$$

According to a result of Hadeler[3] (ref. Remark 7 of [2]), Problem A-1 has a solution $c=(c_1, \cdots, c_n)^T\in\mathsf{R}^n$ if $\max\{r_0, r_1, \cdots, r_k\}=1$ and $\{A, \lambda\}\in\mathscr{X}^*$. Observe that $\mathscr{X}^*$ is a nonempty open set of the Euclidean space $\mathscr{X}$ and so meas $\mathscr{X}^*>0$ if $\max\{r_0, r_1, \cdots, r_k\}=1$. Hence the inequality (2.2) is not only a sufficient but also a necessary condition for the unsolvability of Problem A-1 a.e. ▮

Proof of Theorem 2.3.

1) Suppose that Problem G-2 is solvable at $A, A_1, \cdots, A_m\in\mathsf{R}^{n\times n}$ and $\lambda\in\mathsf{R}^k$. Then there exist nonsingular matrices $Y,\ Z\in\mathsf{R}^{n\times n}$ and a vector $c=(c_1, \cdots, c_m)^T\in\mathsf{R}^m$ such that

$$A+\sum_{i=1}^m c_iA_i=Y\,\mathrm{diag}\,(O^{(r_0)}, \lambda_1I^{(r_1)}, \cdots, \lambda_kI^{(r_k)})Z^T, \tag{3.18}$$

where

$$A=(a_{ij}),\ A_i=(a_{ij}^{(i)}),\ i=1, \cdots, m,\ Y=(y_1, \cdots, y_n),\ Z=(z_1, \cdots, z_n) \tag{3.19}$$

and

$$Z^TY=I^{(n)}. \tag{3.20}$$

Without loss of generality we may assume that $r_0=\max\{r_0, r_1, \cdots, r_m\}$ and

$$y_i^Ty_i=1, \quad i=1, \cdots, n. \tag{3.21}$$

Let

$$a=(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{n1}, a_{n2}, \cdots, a_{nn})^T\in\mathsf{R}^{n^2}, \tag{3.22}$$

$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in\mathsf{R}^k, \quad c=(c_1, \cdots, c_m)^T\in\mathsf{R}^m, \tag{3.23}$$

$$y=(y_{r_0+1}^T, \cdots, y_n^T)^T, \quad z=(z_{r_0+1}^T, \cdots, z_n^T)^T\in\mathsf{R}^{n(n-r_0)} \tag{3.24}$$

and

$$\mathscr{E}=\underbrace{\mathbb{R}^{n\times n}\times\cdots\times R^{n\times n}}_{m+1}\times\mathbb{R}^{k}\times\mathbb{R}^{m}\times\mathbb{R}^{n(n-r_0)}\times\mathbb{R}^{n(n-r_0)}.$$

We define differentiable real-valued functions $g_{ij}(1\leqslant i,\ j\leqslant n)$, $h_{ij}(1\leqslant i,\ j\leqslant n-r_0)$ and $l_i(1\leqslant i\leqslant n-r_0)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{ij})=A+\sum_{t=1}^{m}c_tA_t-(y_{r_0+1},\ \cdots,\ y_n)\operatorname{diag}(\lambda_1I^{(r_1)},\ \cdots,\ \lambda_kI^{(r_k)})(z_{r_0+1},\ \cdots,\ z_n)^T,$$

$$(h_{ij})=(z_{r_0+1},\ \cdots,\ z_n)^T(y_{r_0+1},\ \cdots,\ y_n)-I^{(n-r_0)},$$

$$l_i=y_{r_0+i}^Ty_{r_0+i}-1,\quad i=1,\ \cdots,\ n-r_0.$$

Set

$$g=(g_{11},\ g_{12},\ \cdots,\ g_{1n},\ g_{21},\ g_{22},\ \cdots,\ g_{2n},\ \cdots,\ g_{n1},\ g_{n2},\ \cdots,\ g_{nn}),$$

$$h=(h_{11},\ h_{12},\ \cdots,\ h_{1,n-r_0},\ h_{21},h_{22},\ \cdots,\ h_{2,n-r_0},\ \cdots,\ h_{n-r_0,1},\ h_{n-r_0,2},\ \cdots,\ h_{n-r_0,n-r_0}),$$

$$l=(l_1,\ l_2,\ \cdots,\ l_{n-r_0})$$

and

$$f=(g,\ h,\ l).$$

Let $\mathfrak{M}\subset\mathscr{E}$ be the set of points $X=\{A,\ A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ y,\ z\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A,\ A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ y,\ z\}\in\mathscr{E}$ is associated a vector

$$x=(a^T,\ a_1^T,\ \cdots,\ a_m^T,\ \lambda^T,\ c^T,\ y^T,\ z^T)^T\in\mathbb{R}^{n^2(m+1)+k+m+2n(n-r_0)},$$

where $A,\ A_1,\ \cdots,\ A_m,\ a,\ a_1,\ \cdots,\ a_m,\ \lambda,\ c,\ y,\ z$ are represented by (3.19) and (3.22)—(3.24). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial g}{\partial a} & \frac{\partial h}{\partial a} & \frac{\partial l}{\partial a}\\ \frac{\partial g}{\partial\tilde a} & \frac{\partial h}{\partial\tilde a} & \frac{\partial l}{\partial\tilde a}\\ \frac{\partial g}{\partial\lambda} & \frac{\partial h}{\partial\lambda} & \frac{\partial l}{\partial\lambda}\\ \frac{\partial g}{\partial c} & \frac{\partial h}{\partial c} & \frac{\partial l}{\partial c}\\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial l}{\partial y}\\ \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial l}{\partial z}\end{pmatrix}=\begin{pmatrix}I^{(n^2)} & 0 & 0\\ * & 0 & 0\\ * & 0 & 0\\ * & 0 & 0\\ * & H_y & L_y\\ * & H_z & L_z\end{pmatrix},$$

where

$$\tilde a=(a_1^T,\ \cdots,\ a_m^T)^T\in\mathbb{R}^{mn^2},$$

$$H_y=\frac{\partial h}{\partial y}=\begin{pmatrix}z_{r_0+1} & 0 & \cdots & 0 & 0 & z_{r_0+2} & 0 & \cdots & 0 & 0 & \cdots & z_n & 0 & \cdots & 0 & 0\\ 0 & z_{r_0+1} & \cdots & 0 & 0 & 0 & z_{r_0+2} & \cdots & 0 & 0 & \cdots & 0 & z_n & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & z_{r_0+1} & 0 & 0 & 0 & \cdots & z_{r_0+2} & 0 & \cdots & 0 & 0 & \cdots & z_n & 0\\ 0 & 0 & \cdots & 0 & z_{r_0+1} & 0 & 0 & \cdots & 0 & z_{r_0+2} & \cdots & 0 & 0 & \cdots & 0 & z_n\end{pmatrix},\tag{3.25}$$

$$H_z=\frac{\partial h}{\partial z}=\begin{pmatrix} y_{r_0+1} & y_{r_0+2} & \cdots & y_n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & y_{r_0+1} & y_{r_0+2} & \cdots & y_n & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & y_{r_0+1} & y_{r_0+2} & \cdots & y_n \end{pmatrix}, \tag{3.26}$$

$$L_y=\frac{\partial l}{\partial y}=\begin{pmatrix} 2y_{r_0+1} & 0 & \cdots & 0 & 0 \\ 0 & 2y_{r_0+2} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2y_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 2y_n \end{pmatrix} \tag{3.27}$$

and

$$L_z=\frac{\partial l}{\partial z}=0. \tag{3.28}$$

Let

$$G=\begin{pmatrix} H_y & L_y \\ H_z & L_z \end{pmatrix}\in \mathbb{R}^{2n(n-r_0)\times(n-r_0)(n-r_0+1)}. \tag{3.29}$$

It is easy to see that for any vector $w\in\mathbb{R}^{(n-r_0)(n-r_0+1)}$ from $Gw=0$ one can deduce $w=0$. Therefore we have

$$\operatorname{rank}(G)=(n-r_0)(n-r_0+1).$$

Hence for each point $X\in\mathfrak{M}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at X has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=n^2+(n-r_0)(n-r_0+1).$$

By Theorem 3.1, it follows from

$$\dim(\mathscr{E})=n^2(m+1)+k+m+2n(n-r_0)$$

that $\mathfrak{M}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=mn^2+k+m+(n-r_0)(n+r_0-1). \tag{3.30}$$

Let

$$\mathscr{X}=\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m+1}\times\mathbb{R}^k,$$

and let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A, A_1, \cdots, A_m, \lambda\}\in\mathscr{X}$ at which Problem G-2 is solvable. We define a differentiable mapping F of $\mathfrak{M}\to\mathscr{X}$:

$$F(X)=\{A, A_1, \cdots, A_m, \lambda\}\in\mathscr{X} \quad \text{for} \quad X=\{A, A_1, \cdots, A_m, \lambda, c, y, z\}\in\mathfrak{M},$$

and write

$$\mathfrak{M}'=F(\mathfrak{M}).$$

Since

$$\dim(\mathscr{X})=n^2(m+1)+k,$$

and $\dim(\mathfrak{M})<\dim(\mathscr{X})$ if

$$n-m+r_0(r_0-1)>0, \tag{3.31}$$

by Lemma 3.1 the set $\mathfrak{M}'$ has measure zero under the assumption of (3.31). Observe that for any point $X^{(1)}=\{A, A_1, \cdots, A_m, \lambda\}\in\mathscr{L}$ there exist $c\in\mathsf{R}^m$ and $y, z\in\mathsf{R}^{n(n-r_0)}$ such that the point $X=\{A, A_1,\cdots, A_m, \lambda, c, y, z\}\in\mathfrak{M}$. Hence $\mathscr{L}\subset\mathfrak{M}'$, and thus the set $\mathscr{L}$ has measure zero in the space $\mathscr{X}$. This means that Problem G–2 is u.s.a.e. if the condition (2.3) is fulfilled.

2) We consider the case of $m=n$. Let

$$\mathscr{X}=\underbrace{\mathsf{R}^{n\times n}\times\cdots\times\mathsf{R}^{n\times n}}_{n+1}\times\mathsf{R}^n$$

and

$$\mathscr{X}_s=\{\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}: A_i=(a_{ij}^{(t)}),\ 1<a_{ii}^{(i)}<2, |a_{ii}^{(t)}|<s,\ i\neq t,\ 1\leqslant i,\ t\leqslant n\},$$

where s is a fixed positive number satisfying $s\ll\frac{1}{n-1}$. Obviously, $\mathscr{X}_s$ is an open set of the Euclidean space $\mathscr{X}$, and the matrix

$$E=(a_{ii}^{(j)})_{i,j=1,\cdots,n}$$

is nonsingular provided $\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s$. Let

$$E^{-1}=(l'_{ij}),\quad \mu=\sup_{\substack{\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s\\ 1\leqslant i, j\leqslant n}}|l'_{ij}|,$$

$$\widetilde{A}_t=\sum_{i=1}^{n} l'_{ti}A_i,\quad t=1, \cdots, n$$

and

$$S=\sum_{t=1}^{n}|A_t|,\ \widetilde{S}=\sum_{t=1}^{n}|\widetilde{A}_t|\quad \forall\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s,$$

$$d(\lambda)=\min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}|\lambda_i-\lambda_j|\quad \forall\ \lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathsf{R}^n,$$

$$a_0=(a_{11}, \cdots, a_{nn})^T\in\mathsf{R}^n\quad \forall A=(a_{ij})\in\mathsf{R}^{n\times n},$$

$$k^{(1)}=n\mu\|\lambda-a_0\|_\infty k_1(S)+k_1(A)\quad \forall\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s$$

and

$$\mathscr{X}^*=\left\{\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}:\ k^{(1)}>0,\ d(\lambda)>\frac{4k^{(1)}}{1-n\mu k_1(S)}>0\right\}.$$

From

$$k_1(\widetilde{S})\leqslant n\mu k_1(S)\quad \forall\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s$$

we see that if $r=1$ and $\{A, A_1, \cdots, A_n, \lambda\}\in\mathscr{X}^*$ then by Theorem 1 of [2] Problem G–2 has a solution $c=(c_1, \cdots, c_n)^T\in\mathsf{R}^n$. Observe that $\mathscr{X}^*$ is a nonempty open set of $\mathscr{X}$ and so meas $\mathscr{X}^*>0$ if $r=1$. Hence in the case of $m=n$ the inequality $r>1$ is not only a sufficient but also a necessary condition for the unsolvability of Problem G–2 a.e. ∎

Proof of Theorem 2.4.

1) Suppose that Problem A–2 is solvable at $A\in$ [illegible] and $\lambda\in\mathsf{R}^k$, then there exist nonsingular matrices Y, $Z\in\mathsf{R}^{n\times n}$ and a vector $c=(c_1, \cdots, c_n)^T\in\mathsf{R}^n$ such that

$$A+\mathrm{diag}(c_1, \cdots, c_n)=Y\,\mathrm{diag}(\text{[illegible]}, \cdots, \lambda_k I^{(r_k)})Z^T, \tag{3.32}$$

where

$$A=(a_{ij}),\quad Y=(y_1,\cdots,y_n),\quad Z=(z_1,\cdots,z_n) \tag{3.33}$$

and

$$Z^TY=I^{(n)},\quad y_i^Ty_i=1,\quad i=1,\cdots,n. \tag{3.34}$$

Without loss of generality we may assume that $r_0=\max\{r_0, r_1,\cdots,r_k\}$ and $A\in R_0^{n\times n}$.

Let

$$a=(a_{12},a_{13},\cdots,a_{1n},a_{21},a_{23},\cdots,a_{2n},\cdots,a_{n1},a_{n2},\cdots,a_{n,n-1})^T\in R^{n(n-1)}, \tag{3.35}$$

$$\lambda=(\lambda_1,\cdots,\lambda_k)^T\in R^k, \tag{3.36}$$

$$y=(y_{r_0+1}^T,\cdots,y_n^T)^T,\quad z=(z_{r_0+1}^T,\cdots,z_n^T)^T\in R^{n(n-r_0)} \tag{3.37}$$

and

$$\mathscr{E}=R_0^{n\times n}\times R^k\times R^{n(n-r_0)}\times R^{n(n-r_0)}.$$

We define differentiable real-valued functions $g_{ij}(1\leqslant i,\ j\leqslant n)$, $h_{ij}(1\leqslant i,\ j\leqslant n-r_0)$ and $l_i(1\leqslant i\leqslant n-r_0)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{ij})=A-(y_{r_0+1},\cdots,y_n)\operatorname{diag}(\lambda_1I^{(r_1)},\cdots,\lambda_kI^{(r_k)})(z_{r_0+1},\cdots,z_n)^T,$$

$$(h_{ij})=(z_{r_0+1},\cdots,z_n)^T\ (y_{r_0+1},\cdots,y_n)-I^{(n-r_0)},$$

$$l_i=y_{r_0+i}^Ty_{r_0+i}-1,\quad i=1,\cdots,n-r_0.$$

Set

$$g=(g_{12},g_{13},\cdots,g_{1n},g_{21},g_{23},\cdots,g_{2n},\cdots,g_{n1},g_{n2},\cdots,g_{n,n-1}),$$

$$h=(h_{11},h_{12},\cdots,h_{1,n-r_0},h_{21},h_{22},\cdots,h_{2,n-r_0},h_{n-r_0,1},h_{n-r_0,2},\cdots,h_{n-r_0,n-r_0}),$$

$$l=(l_1,l_2,\cdots,l_{n-r_0})$$

and

$$f=(g,h,l).$$

Let $\mathfrak{M}\subset\mathscr{E}$ be the set of points $X=\{A,\lambda,y,z\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A,\lambda,y,z\}\in\mathscr{E}$ is associated a vector

$$x=(a^T,\lambda^T,y^T,z^T)^T\in R^{n(n-1)+k+2n(n-r_0)},$$

where A, a, λ, y, z are represented by (3.33) and (3.35)—(3.37). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial g}{\partial a} & \frac{\partial h}{\partial a} & \frac{\partial l}{\partial a}\\ \frac{\partial g}{\partial\lambda} & \frac{\partial h}{\partial\lambda} & \frac{\partial l}{\partial\lambda}\\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial l}{\partial y}\\ \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial l}{\partial z}\end{pmatrix}=\begin{pmatrix}I^{(n(n-1))} & 0 & 0\\ * & 0 & 0\\ * & H_y & L_y\\ * & H_z & L_z\end{pmatrix},$$

where H_y, H_z, L_y and L_z are the matrices denoted by (3.25)—(3.28), and

$$\operatorname{rank}\begin{pmatrix}H_y & L_y\\ H_z & L_z\end{pmatrix}=(n-r_0)(n-r_0+1).$$

Hence for each point $X\in\mathfrak{M}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at X has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=n(n-1)+(n-r_0)(n-r_0+1).$$

By Theorem 3.1, it follows from

$$\dim(\mathscr{E})=n(n-1)+k+2n(n-r_0)$$

that $\mathfrak{M}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=k+(n-r_0)(n+r_0-1). \tag{3.38}$$

Let

$$\mathscr{X}=\mathsf{R}_0^{n\times n}\times\mathsf{R}^k,$$

and let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A,\lambda\}\in\mathscr{X}$ at which Problem A-2 is solvable. We define a differentiable mapping F of $\mathfrak{M}\to\mathscr{X}$:

$$F(X)=\{A,\lambda\}\in\mathscr{X}\quad\text{for}\quad X=\{A,\lambda,y,z\}\in\mathfrak{M},$$

and write

$$\mathfrak{M}'=F(\mathfrak{M}).$$

Since

$$\dim(\mathscr{X})=n(n-1)+k,$$

and $\dim(\mathfrak{M})<\dim(\mathscr{X})$ if

$$r_0>1, \tag{3.39}$$

by Lemma 3.1 the set $\mathfrak{M}'$ has measure zero under the assumption of (3.39). Observe that for any point $X^{(1)}=\{A,\lambda\}\in\mathscr{L}$ there exist $y,z\in\mathsf{R}^{n(n-r_0)}$ such that the point $X=\{A,\lambda,y,z\}\in\mathfrak{M}$. Hence $\mathscr{L}\subset\mathfrak{M}'$, and thus the set $\mathscr{L}$ has measure zero in the space $\mathscr{X}$. This means that Problem A-2 is u.s.a.e. if the condition (2.4) is fulfilled.

2) Let

$$\mathscr{X}=\mathsf{R}_0^{n\times n}\times\mathsf{R}^n,$$

$$d(\lambda)=\min_{\substack{1\le i,j\le n\\ i\ne j}}|\lambda_i-\lambda_j|\quad\forall\lambda=(\lambda_1,\cdots,\lambda_n)^T\in\mathsf{R}^n$$

and

$$\mathscr{X}^*=\{\{A,\lambda\}\in\mathscr{X}: d(\lambda)>4k_1(A)\}.$$

According to Remark 3 of [2], Problem A-2 has a solution $c=(c_1,\cdots,c_n)^T\in\mathsf{R}^n$ if $\max\{r_0,r_1,\cdots,r_k\}=1$ and $\{A,\lambda\}\in\mathscr{X}^*$. Observe that $\mathscr{X}^*$ is a nonempty open set of the Euclidean space $\mathscr{X}$ and so meas $\mathscr{X}^*>0$ if $\max\{r_0,r_1,\cdots,r_k\}=1$. Hence the inequality (2.4) is not only a sufficient but also a necessary condition for the unsolvability of Problem A-2 a.e. ∎

§ 4. Final Remarks

4.1. We consider a simple example:

Problem A-1(3, 2, 1) Given a non-zero $\lambda_1\in\mathsf{R}$ and a matrix $A=(a_{ij})\in S\mathsf{R}_0^{3\times 3}$, find a vector $c=(c_1,c_2,c_3)^T\in\mathsf{R}^3$ such that the matrix $A+\operatorname{diag}(c_1,c_2,c_3)$ has zero eigenvalue of multiplicity 2 and a simple eigenvalue λ_1.

By the definition on the unsolvability a.e. given by Shapiro [5] and Theorem 1.1, from

$$\frac{r_0(r_0+1)}{2}+\frac{r_1(r_1+1)}{2}\not> m+k \text{ (here } m=3,\ k=1,\ r_0=2,\ r_1=1)$$

it is unable to determine whether or not Problem A-1 (3; 2, 1) is u.s.a.e. But by our Theorem 2.2, since $r_0>1$, Problem A-1 (3; 2, 1) is u.s.a.e. according to Definition 1.2.

It is easy to prove that Problem A-1 (3; 2, 1) is solvable if and only if

(i) $a_{12}=a_{13}=a_{23}=0$

or

(ii) $a_{12}=a_{13}=0,\ |a_{23}|\leqslant\frac{1}{2}|\lambda_1|$

or

(iii) $a_{12}=a_{23}=0,\ |a_{13}|\leqslant\frac{1}{2}|\lambda_1|$

or

(iv) $a_{13}=a_{23}=0,\ |a_{12}|\leqslant\frac{1}{2}|\lambda_1|$

or

(v) $a_{12}a_{13}a_{23}\neq 0,\ \frac{a_{12}a_{13}}{a_{23}}+\frac{a_{12}a_{23}}{a_{13}}+\frac{a_{13}a_{23}}{a_{12}}=\lambda_1.$

The conditions (i)—(v) show that Problem A-1 (3; 2, 1) is solvable only in a lower dimensional set of the space $SR_0^{3\times3}\times\mathbf{R}$. Hence it is really u.s.a.e.

4.2. It is easy to see from Theorem 2.1 that Problem G-1 is u.s.a.e. if the natural numbers m, k and the nonnegative integers $r_0, r_1, \cdots, r_k$ satisfy Shapiro's condition (1.1).

4.3. Wang and Garbow [7] has discussed a numerical method for solving Problem G-1 in which $m\leqslant n$. Theorem 2.1 shows that Problem G-1 is u.s.a.e. if $m<n$. Hence, in a general way, it is impossible to solve Problem G-1 in the case $m<n$ unless we treat this problem in other senses (e.g., in the sense of least squares approximation).

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