THE USE OF MAXIMAL MONOTONE OPERATORS IN THE NUMERICAL ANALYSIS OF VARIATIONAL INEQUALITIES AND FREE BOUNDARY PROBLEMS*

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Abstract

In this work we use maximal monotone operators theory for solving some variational inequalities and some free boundary problems. This utilization, already adopted at theoretical level for giving existence, uniqueness and regularity theorems ([1], [6], [10]) in this field, was ignored until now in numerical methods. We obtain in this way a direct computation method extremely simple and general enough as we shall prove in concrete examples.

Some Second Order Problem

§ 1. Introduction

We adopt usual notations in the field of variational inequalities ([5], [8]). Let
V be a real Sobolev space,
K a closed convex set in V,
$a(\cdot, \cdot)$ a bilinear, continuous, V-elliptic form on $V\times V$:

$$|a(u, v)|\leqslant M\|u\|\|v\|, \quad \forall u, v\in V,$$
$$a(u, u)\geqslant\alpha\|u\|^2, \ \alpha>0, \quad \forall u\in V,$$

$\langle\cdot, \cdot\rangle$ the duality pairing between V' and V,

χ: $V\to R$ a proper convex function, lower-semicontinuous, on V, having K as effective domain,

$f\in V'$ an assigned function, which we suppose belongs to $L^2(\Omega)$ in numerical examples.

We consider the following variational inequality:

$$\text{to find } u\in V: a(u, v-u)+\chi(v)-\chi(u)\geqslant\langle f, v-u\rangle, \ \forall v\in V. \tag{1.1}$$

It is well known that (1.1) has a unique solution ([8]).

We recall that a vector $\mu(u)$ is said to be a subgradient of χ at a point u, if $\mu(u)$ satisfies the "subgradient inequality":

$$\chi(v)-\chi(u)\geqslant\langle\mu(u), v-u\rangle, \ \forall v\in V. \tag{1.2}$$

The subdifferential $\partial\chi(u)$, a multivalued mapping from V to $2^{V'}$, is the set of all subgradients ([7]). We suppose $\partial\chi(v)\neq\emptyset$, $\forall v\in K$; (0, 0) belongs to the graph of $\partial\chi$. If we consider the operator A: $V\to V'$ associated with $a(\cdot, \cdot)$:

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$$\langle Au, v\rangle = a(u, v), \tag{1.3}$$

we obtain from the theory of maximal monotone operators ([1], [11]) that A and $\partial\chi$ as well as $A+\partial\chi$ are maximal monotone operators. Besides, $A+\partial\chi$ is coercive. Thus the problem:

$$\{u, \mu(u)\}\in V\times V': \langle Au+\mu(u), v\rangle = \langle f, v\rangle, \quad \forall v\in V \tag{1.4}$$

has a solution ([11], Corollary, p. 120) and only one solution. In fact, let u_1, u_2 be two solutions; by setting $u=u_1$, $v=u_1-u_2$ in (1.4) and then $u=u_2$, $v=u_2-u_1$, we obtain respectively

$$\langle Au_1+\mu(u_1), u_1-u_2\rangle = \langle f, u_1-u_2\rangle, \tag{1.5}$$

$$\langle Au_2+\mu(u_2), u_2-u_1\rangle = \langle f, u_2-u_1\rangle. \tag{1.6}$$

By adding (1.5), (1.6) and by using V-ellipticity of A and monotonicity of $\partial\chi$ we have

$$0\leqslant\alpha\|u_1-u_2\|^2\leqslant\langle A(u_1-u_2), u_1-u_2\rangle+\langle\mu(u_1)-\mu(u_2), u_1-u_2\rangle=0 \tag{1.7}$$

which proves that $u_1=u_2$. $\mu(u)$ is a section of $\partial\chi$. (1.4) is equivalent to (1.1). The same type of equation appears in some free boundary problems, for example in Stefan's problem.

§ 2. Two Obstacles Problem

Let

Ω be an open, bounded set in R^n with boundary $\partial\Omega=\Gamma$ regular enough. Ω is a convex polygon in R^2, in numerical examples,

$V=H_0^1(\Omega)$,

$a(u, v)=\int_\Omega \text{grad}\, u\times \text{grad} v\cdot dx,\ Au=-\Delta u,$

$K_\alpha^\beta=\{v: v\in V/\beta\geqslant v\geqslant\alpha$ a.e. in $\Omega\}$. α and β are assigned functions such that $\beta|_\Gamma\geqslant 0\geqslant\alpha|_\Gamma$; we can suppose α, $\beta\in C^0(\bar\Omega)$ and $\alpha=0$ by translation.

We consider the variational inequality ([5], Chapter II, Section 6):

$$u\in K_\alpha^\beta: a(u, v-u)\geqslant\langle f, v-u\rangle, \quad \forall v\in K_\alpha^\beta, \tag{2.1}$$

and consequently have as "indicator" function of K_α^β

$$\chi(v)\begin{cases}=0 & \text{if } v\in K_\alpha^\beta,\\ =+\infty & \text{if } v\notin K_\alpha^\beta\end{cases} \tag{2.2}$$

and then like subdifferential:

$$\partial_\chi(v)\begin{cases}=\emptyset & \text{if } v\notin K_\alpha^\beta,\\ =]-\infty,0] & \text{if } v=\alpha,\\ =0 & \text{if } \beta>v>\alpha,\\ =[0,+\infty[& \text{if } v=\beta.\end{cases}$$

(2.1) is equivalent to

$$u\in V: a(u, v-u)+\chi(v)-\chi(u)\geqslant\langle f, v-u\rangle, \quad \forall v\in V, \tag{2.4}$$

$$\{u, \mu(u)\}\in V\times V': a(u, v)+\langle\mu(u), v\rangle=\langle f, v\rangle, \quad \forall v\in V. \tag{2.5}$$

We must determine, besides u, a measure $\mu(u)=\mu^+-\mu^-$ satisfying (2.5).

Discrete Problem

With the aim of discretizing (2.5) we construct two finite-dimensional spaces V_h, V'_h respectively contained in V and in V'. We consider first a "regular" triangulation T_h of Ω such that all triangles $T\in T_h$ have all angles$\leqslant\pi/2$ ([2]) and denote by $I_0=\{1, \cdots, N_0\}$ the set of indexes related with internal nodes x_i of T_h, by $I_\partial=\{N_0+1, \cdots, N\}$ the set related to boundary nodes, by I the set $I_0 U I_\partial$. Let Σ_h be the set of sides x_rx_s $(r, s\in I)$ of triangles $T\in T_h$, $l_{r,s}$ their length. $V_h=H^1_{0,h}(\Omega)$ is the space $\{\phi^h_i(x)\}_{I_0}$ spanned by finite affine elements $\phi^h_i(x)$, $i\in I_0$. Then by using Green's formula we can verify that the measure $\mu_h=Au_h\in V'$ of $u_h\in V_h$ consists of Dirac measures $\delta^h_{r,s}$ concentrated on x_rx_s:

$$\langle\mu_h, v_h\rangle=\int_\Omega v_h d\mu_h=\sum_{r,s}M_{r,s}\int_{x_rx_s}v_h ds, \quad M_{r,s}\in R, \ \forall v_h\in V_h. \tag{2.6}$$

In particular we have

$$\langle\mu_h, \phi^h_i\rangle=\sum_{s=1}^{k_i}M_{i,s}\int_{x_ix_s}\phi^h_i ds=1/2\sum_{s=1}^{k_i}M_{i,s}l_{i,s}=M_i, \quad i\in I_0,$$

where $x_s(s=1, \cdots, k_i)$ is a node contiguous to x_i in the support ω_i of ϕ^h_i. To an assigned function $u_h\in V_h$, owing to linear independence of $\delta^h_{r,s}$ there corresponds a unique determination of the coefficients $M_{r,s}$ and thus of $M=\{M_i\}_{I_0}$.

On the other hand, denoting $\{a'_{i,j}\}_{I_0\times I_0}$ the inverse of matrix $\{a(\phi^h_j, \phi^h_i)\}_{I_0\times I_0}$, $M=\{M_i\}_{I_0}$ and Σ_h determine the function $u_h=\sum_{i,j\in I_0}M_i a'_{i,j}\phi^h_j$ such that

$$a(u_h, \phi^h_i)=M_i, \quad i\in I_0. \tag{2.7}$$

$V'_h=H^{-1}_h(\Omega)$ is the space spanned by Dirac measures $\delta^h_{r,s}$ located on Σ_h. It is well known ([5]) that V_h is an approximation of V.

By using the mentioned spaces, we obtain the discrete problem:

$$\{u_h, u_h(\mu_h)\}\in V_h\times V'_h\text{: } a(u_h, v_h)+\langle\mu_h(u_h), v_h\rangle=\langle f, v_h\rangle, \quad \forall v_h\in V_h \tag{2.8}$$

and thus the algebraic system:

$$\{U, M(U)\}\in R^{N_0}\times R^{N_0}\text{: } AU+M=b, \tag{2.9}$$

where $U=\{U_i\}_{I_0}$,

$A=\{a_{i,j}\}_{I_0\times I_0}=\{a(\phi^h_j, \phi^h_i)\}_{I_0\times I_0}$ "stiffness matrix",

$b=\{b_i\}_{I_0}=\{(f, \phi^h_i)_{L^2(\Omega)}\}_{I_0}$ "load term".

Algorithm

We solve (2.9) by using a Jacobi type iterative method, like

$$a_{i,i}U^{n+1}_i+M^{n+1}_i=G_i(U^n), \quad i\in I_0, \tag{2.10}$$

where $G_i(U)=b_i-\sum_{\substack{j\in I_0\\ j\neq i}}a_{i,j}U_j$.

We set $A_i=\alpha(x_i)$, $B_i=\beta(x_i)$ and consider the following cases:

I) if $G_i(U^n)\leqslant a_{i,i}A_i$, we calculate $M^{n+1}_i\leqslant 0$ such that

$$a_{i,i}A_i+M^{n+1}_i=G_i(U^n). \tag{2.11}$$

We have contact with lower obstacle in x_i and a negative measure μ_h on ω_i;

II) if $a_{ii}A_i<G_i(U^n)<a_{i,i}B_i$, we estimate U^{n+1} such that

$$a_{i,i}U^{n+1}_i=G_i(U^n). \tag{2.12}$$

We have detachment from obstacles in x_i, μ_h is zero in ω_i;

III) if $G_i(U^n) \geqslant a_{i,i} B_i$, we calculate $M_i^{n+1} \geqslant 0$ such that

$$a_{i,i} B_i + M_i^{n+1} = G_i(U^n). \tag{2.13}$$

We have contact with upper obstacle and a positive measure on ω_i.

Convergence of the algorithm

By confronting (2.10) and (2.9) we have

$$a_{ii}(U_i - U_i^{n+1}) + M_i - M_i^{n+1} = G_i(U) - G_i(U^n), \quad i \in I_0. \tag{2.14}$$

At this point we emphasize the monotonicity properties of operator $\partial\chi_h$ and $A(a_{i,i} > 0,\ a_{i,j} \leqslant 0,\ i \neq j,\ A^{-1} \geqslant 0)$ (see [2] p. 172, [13] p. 85, Corollary 2; p. 87, Ex. 7). Then we can easily verify the following relation:

$$\operatorname{sign}(U_i - U_i^{n+1}) = \operatorname{sign}(M_i - M_i^{n+1}) \tag{2.15}$$

by ascertaining, in all cases I), II), III), that

$$A_i \leqslant U_i^{n+1} \leqslant U_i \leqslant B_i \quad \text{implies } M_i^{n+1} \leqslant M_i. \tag{2.16}$$

From (2.14)—(2.15) we obtain

$$a_{i,i} |U_i - U_i^{n+1}| \leqslant a_{i,i} |U_i - U_i^{n+1}| + |M_i - M_i^{n+1}| \leqslant \sum_{\substack{j \in I_0 \\ j \neq i}} |a_{i,j}| \cdot |U_j - U_j^n| \tag{2.17}$$

and then

$$\|U - U^{n+1}\|_\infty \leqslant \rho(|a_{i,j}|/a_{i,i}) \|U - U^n\|_\infty, \tag{2.18}$$

where $\|\cdot\|_\infty$ is l_∞ norm,

ρ, the spectral radius of matrix $\{|a_{i,j}|/a_{i,i}\}$, is <1 (see [13], Th. 3.13).

Thus $\{U^n, M^n\}_n$ converges to solution $\{U, M\}$ of (2.9).

§ 3. The Semipermeable Membrane Problem

Let

$V = H^1(\Omega)$,

$a(u, v) = \int_\Omega (\operatorname{grad} u \times \operatorname{grad} v + uv) \cdot dx$,

$K = \{v: v \in V / v \geqslant 0 \text{ a.e. on } \Gamma\}$.

We consider the variational inequality ([5], Chapter 4, Section 3):

$$u \in K: a(u, v-u) \geqslant \langle f, v-u \rangle, \quad \forall v \in K; \tag{3.1}$$

the indicator function of K

$$\chi(v) \begin{cases} = 0 & \text{if } v \geqslant 0, \text{ on } \Gamma, \\ = +\infty & \text{if } v < 0 \text{ on } \Gamma \end{cases} \tag{3.2}$$

and the subdifferential

$$\partial\chi(v) \begin{cases} = \emptyset & \text{if } v < 0, \\ =]-\infty, 0] & \text{if } v = 0, \\ = 0 & \text{if } v > 0 \text{ on } \Gamma. \end{cases} \tag{3.3}$$

(3.1) is equivalent to the following problems:

$$u \in V: a(u, v-u) + \chi(v) - \chi(u) \geqslant \langle f, v-u \rangle, \quad \forall v \in V; \tag{3.4}$$

$$\{u, \mu(u)\} \in V \times H^{-1/2}(\Gamma): a(u, v) + (\mu(u), v) = \langle f, v \rangle, \ \forall v \in V, \tag{3.5}$$

where $(\cdot, \cdot)$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ and $\mu = -\gamma\mu$.

The Discrete Problem

We consider now a "regular" triangulation such that all triangles $T\in T_h$ have all angles $\leqslant\theta_0<\pi/2$ ([2] p. 172) and the set I) of Section 2, and construct the corresponding spaces $V_h=H_h^1(\Omega)=\{\phi_i^h\}_I$, $V_h'=\{\delta_{r,s}\}$. By operating the discretization of (3.5) we obtain

$$\{u_h,\ \mu_h(u_h)\}\in V_h\times V_h':\ a(u_h,\ v_h)+\langle\!\langle\mu_h(u_h),\ v_h\rangle\!\rangle=\langle f,\ v_h\rangle,\quad \forall v_h\in V_h \tag{3.6}$$

and the algebraic system

$$(A+h^2C)U+M=b, \tag{3.7}$$

where $U=\{U_j\}_I$,

$M=\{M_j\}_I=\{O_r,\ M_i\}_{I_0\times I_2}$,

$A=\{a_{i,j}\}_{I\times I}=\{(\text{grad}\ \phi_j^h,\ \text{grad}\ \phi_i^h)_{L^2(\Omega)}\}_{I\times I}$,

$C=\{c_{i,j}\}_{I\times I}=\{1/h^2(\phi_j^h,\ \phi_i^h)_{L^2(\Omega)}\}_{I\times I}$ is the positive "mass matrix".

Algorithm

We more explicitly write the system (3.7) as follows:

$$\sum_{j\in I}(a_{i,j}+h^2c_{i,j})U_j=b_i,\quad i\in I_0, \tag{3.8}$$

$$\sum_{j\in I}(a_{i,j}+h^2c_{i,j})U_j+M_i=b_i,\quad i\in I_2 \tag{3.9}$$

and put

$$G_i(U)=b_i-\sum_{\substack{j\in I\\ j\neq i}}(a_{i,j}+h^2c_{i,j})U_j. \tag{3.10}$$

Then we calculate U^{n+1} and M^{n+1} such that

$$(a_{i,i}+h^2c_{i,i})U_i^{n+1}=G_i(U^n)\quad \text{if } i\in I_0, \tag{3.11}$$

$$M_i^{n+1}=G_i(U^n)\quad \text{if } G_i(U^n)\leqslant 0,\ i\in I_2, \tag{3.12}$$

$$(a_{i,i}+h^2c_{i,i})U^{n+1}=G_i(U^n)\quad \text{if } G_i(U^n)>0,\ i\in I_2. \tag{3.13}$$

We can verify the convergence of the iterative method as in the previous section.

§ 4. The Elasto-Plastic Torsion of a Bar

Let

V, $a(\ ,\ )$ be like in Section 2,

$L=L^\infty(\Omega)$, $L'=$space of Radon's measures,

$\Lambda=\{p:\ p\in L/p\geqslant 0 \text{ a.e. in } \Omega\}$,

$\psi_p(u)=1/2\int_\Omega p\cdot[|\text{grad}\ u|^2-1]\cdot dx,\quad p\in\Lambda.$

We consider the variational inequality ([5], Chapter 3):

$$u\in K;\ a(u,\ v-u)\geqslant\langle f,\ v-u\rangle,\quad \forall v\in K, \tag{4.1}$$

where

$$K=\{v:\ v\in V/|\text{grad}\ v|\leqslant 1 \text{ a.e. in } \Omega\}. \tag{4.2}$$

We can define the indicator function of K in the following way:

$$\chi(u)\begin{cases}=0 & \text{if } u\in K,\\ =+\infty & \text{if } u\notin K,\end{cases}=\sup_{p\in\Lambda}\psi_p(u), \tag{4.3}$$

$\psi_p(u)$ is a differentiable function and we have of course

$$\langle d\psi_p(u),\ v\rangle = \int_\Omega p\cdot \mathrm{grad}\ u\cdot \mathrm{grad}\ v\ dx \tag{4.4}$$

and thus, by calculating the subdifferential of (4.3), we find

$$\langle \mu(u),\ v\rangle = \int_\Omega v\, d\mu(u) = \int_\Omega q\cdot \mathrm{grad}\ u\cdot \mathrm{grad}\ v\ dx, \tag{4.5}$$

where

$$q\begin{cases} =0 & \text{if } |\mathrm{grad}\ u|<1,\\ \in [0,\ +\infty[& \text{if } |\mathrm{grad}\ u|=1,\\ \in\emptyset & \text{if } |\mathrm{grad}\ u|>1. \end{cases} \tag{4.6}$$

Instead of (4.1) we obtain the equivalent problem:

$$\{u,\ q\}\in V\times A:\ ((1+q)\ \mathrm{grad}\ u,\ \mathrm{grad}\ v)_{L^2(\Omega)} = \langle f,\ v\rangle,\quad \forall v\in V. \tag{4.7}$$

The Discrete Problem

We set

$$A_h = \{p_h:\ p_h(x) = \sum_{r\in J} c_r(x) p_r,\ p_r\in R^+,\ r\in J\}, \tag{4.8}$$

where $J=\{1,\ \cdots,\ S\}$ is the set of indexes related to triangles $T_r\in T_h$,

$S=\mathrm{card}(\bigcup T_r)$,

$c_r(x)$ is the characteristic function of $\mathring{T}_r$ (the interior of T_r).

We obtain the discrete problem:

$$\{u_h,\ q_h\}\in V_h\times A_h:\ ((1+q_h)\mathrm{grad}\ u_h,\ \mathrm{grad}\ v_h)_{L^2(\Omega)} = (f,\ v_h)_{L^2(\Omega)},\quad \forall v_h\in V_h \tag{4.9}$$

and in particular

$$\{u_h,\ q_h\}\in V_h\times A_h:\ ((1+q_h)\mathrm{grad}\ u_h,\ \mathrm{grad}\ \phi_i^h)_{L^2(\Omega)} = (f,\ \phi_i^h)_{L^2(\Omega)},\quad i\in I_0 \tag{4.10}$$

that is

$$\sum_{r\in J}((1+q_r)\mathrm{grad}\ u_h,\ \mathrm{grad}\ \phi_i^h)_{L^2(T_r)} = (f,\ \phi_i^h)_{L^2(\omega_i)},\quad i\in I_0, \tag{4.11}$$

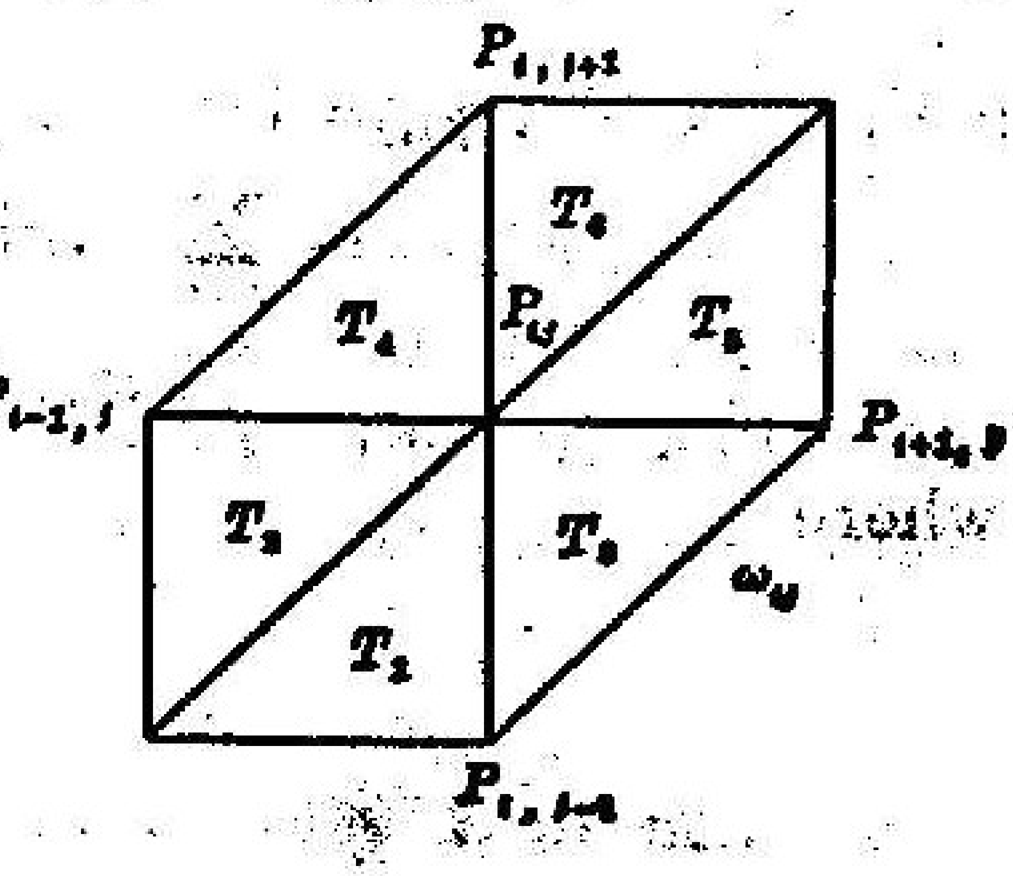
where

$$q_r\begin{cases} =0 & \text{if } |\mathrm{grad}\ u_h|<1 \text{ on } T_r,\\ \in [0,\ +\infty[& \text{if } |\mathrm{grad}\ u_h|=1 \text{ on } T_r. \end{cases} \tag{4.12}$$

The constant term q_r is well fitted for blocking $|\mathrm{grad}\ u_h|$ to maximal unitary value on T_r. We now describe an algorithm for solving (4.11).

Algebraic System

Let us suppose that we use Courant's linear finite elements $\phi_{i,j}^h$ (see figure) related to nodes $P_{i,j}$ and the hexagon $\omega_{i,j}$.



We obtain in this way the following system:

$$\begin{aligned} ((1+q_h)\ \mathrm{grad}\ u_h,\ \mathrm{grad}\ \phi_{i,j}^h)_{L^2(\Omega)} &= -(1+q_{i,j-1})U_{i,j-1} - (1+q_{i-1,j})U_{i-1,j} \\ &\quad +4(1+q_{i,j})U_{i,j} - (1+q_{i+1,j})U_{i+1,j} \\ &\quad -(1+q_{i,j+1})U_{i,j+1} \\ &= b_{i,j} \end{aligned} \tag{4.13}$$

where

$$\begin{cases} q_{i,j-1}=1/2(q_1+q_3), & q_{i-1,j}=1/2(q_2+q_4), \\ q_{i,j}=1/4(q_{i,j-1}+q_{i-1,j}+q_{i+1,j}+q_{i,j+1}), \\ q_{i+1,j}=1/2(q_3+q_5), & q_{i,j+1}=1/2(q_4+q_6). \end{cases} \tag{4.14}$$

We must calculate constants q_r and then $q_{i,j}$ such that the constraint $|\text{grad } u_h|\leqslant 1$ will be satisfied. It is well known that we can realize this aim, step by step, by using Uzawa's algorithm ([5], Chapter 3, Section 3.2.2), that is by putting

$$q_r^n=(q_r^{n-1}+\lambda(|\text{grad } u_h^{n-1}|^2-1)|_{T_r})^+, \quad \lambda>0, \ r\in J \tag{4.15}$$

and then solving the system

$$\begin{aligned} &-(1+q_{i,j-1}^n)U_{i,j-1}^n-(1+q_{i-1,j}^n)U_{i-1,j}^n+4(1+q_{i,j}^n)U_{i,j}^n \\ &-(1+q_{i+1,j}^n)U_{i+1,j}^n-(1+q_{i,j+1}^n)U_{i,j+1}^n=b_{i,j}. \end{aligned} \tag{4.16}$$

We can take as primer function $q_{i,j}^0=0$, thus obtaining as initial configuration $U_{i,j}^0$. By continuing we obtain the sequence $\{q_{i,j}^n, U_{i,j}^n\}_n$ that converges to solution $\{q_{i,j}, U_{i,j}\}$ of (4.13) ([5]).

§ 5. Signorini Problem

We set V and a (,) as in Section 3 and consider the variational inequality ([5], Chapter 4, Section 2)

$$u\in V: \ a(u, v-u)+\chi(v)-\chi(u)\geqslant\langle f, v-u\rangle, \quad \forall v\in V, \tag{5.1}$$

where $\chi(v)$ in this case, is a more regular proper, convex, semicontinuous function (see [11])

$$\chi(v)=\int_\Gamma |v|d\Gamma \text{ (convex integrand).} \tag{5.2}$$

We observe that

$$\partial\chi(v)\begin{cases} =-1 & \text{if } v<0, \\ =[-1, +1] & \text{if } v=0, \\ =+1 & \text{if } v>0 \text{ on } \Gamma \end{cases} \tag{5.3}$$

and then we write the equation equivalent to (5.1) as follows:

$$\{u, \mu(u)\}\in V\times\Lambda: \ a(u, v)+(\mu(u), v)_{L^2(\Gamma)}=(f, v)_{L^2(\Gamma)}, \quad \forall v\in V, \tag{5.4}$$

where

$$\Lambda=\{\mu: \mu\in L^2(\Gamma)/-1\leqslant\mu\leqslant+1 \text{ a.e. on } \Gamma\}.$$

Discrete Problem

By using the notations of Section 3, we put $\mu_h(x)=\sum_{j\in I_2} M_j\phi_j^h(x)$ and calculate the scalar product $\gamma_{i,j}=1/h(\phi_j^h, \phi_i^h)_{L^2(\Gamma)}$. We obtain the following algebraic system:

$$\sum_{j\in I}(a_{i,j}+h^2c_{i,j})U_j=b_i, \quad i\in I_0, \tag{5.5}$$

$$\sum_{j\in I}(a_{i,j}+h^2c_{i,j})U_j+h\sum_{j\in I_2}\gamma_{i,j}M_j=b_i, \quad i\in I_2, \tag{5.6}$$

where

$$M_i\begin{cases} =-1 & \text{if } U_i<0, \\ \in[-1, +1] & \text{if } U_i=0, \\ =+1 & \text{if } U_i>0. \end{cases} \tag{5.7}$$

Algorithm

We set

$$G_i(U^n)=b_i+\sum_{\substack{j\in I\\ j\neq i}}(|a_{i,j}|-h^2c_{i,j})U_j^n,\quad i\in I_0,$$

$$G_i(U^n)=b_i+\sum_{\substack{j\in I\\ j\neq i}}(|a_{i,j}|-h^2c_{i,j})U_j^n-h\sum_{\substack{j\in I_2\\ j\neq i}}\gamma_{i,j}M_j,\quad i\in I_2$$

and propose the following algorithm:

$$(a_{i,i}+h^2c_{i,i})U_i^{n+1}=G_i(U^n),\quad \text{if } i\in I_0, \tag{5.8}$$

$$(a_{i,i}+h^2c_{i,i})U_i^{n+1}-h\gamma_{i,i}=G_i(U^n),\quad \text{if } G_i(U^n)<-h\gamma_{i,i},\ i\in I_2, \tag{5.9}$$

$$h\gamma_{i,i}M_i^{n+1}=G_i(U^n),\quad \text{if } -h\gamma_{i,i}\leqslant G_i(U^n)\leqslant h\gamma_{i,i},\ i\in I_2, \tag{5.10}$$

$$(a_{i,i}+h^2c_{i,i})U_i^{n+1}+h\gamma_{i,i}=G_i(U^n),\quad \text{if } G_i(U^n)>h\gamma_{i,i},\ i\in I_2. \tag{5.11}$$

Convergence of the Algorithm

From (5.5), (5.8), (5.6), (5.9)—(5.11), in the usual way, we obtain

$$|U_i-U_i^{n+1}|\leqslant\sum_{\substack{j\in I\\ j\neq i}}[(|a_{i,j}|-h^2c_{i,j})/(a_{i,i}+h^2c_{i,i})]|U_j-U_j^n|,\quad i\in I_0, \tag{5.12}$$

$$|U_i-U_i^{n+1}|+h\tau_i|M_i-M_i^{n+1}|\leqslant\sum_{\substack{j\in I\\ j\neq i}}\{(|a_{i,j}|-h^2c_{i,j})/(a_{i,i}+h^2c_{i,i})$$

$$\cdot|U_j-U_j^n|+h\sum_{\substack{j\in I_2\\ j\neq i}}\tau_i(\gamma_{i,j}/\gamma_{i,i})|M_j-M_j^n|\},\quad i\in I_2, \tag{5.13}$$

where $\tau_i=\gamma_{i,i}/(a_{i,i}+h^2c_{i,i})$ is positive.

We can unify (5.12), (5.13) by setting $\tau_i=0$ when $i\in I_0$ and

$$N(U-U^n,\ M-M^n)=\sup_{i\in I}(|U_i-U_i^n|+h\tau_i|M_i-M_i^n|). \tag{5.14}$$

From (5.12), (5.13) we have

$$N(U-U^n,\ M-M^{n+1})\leqslant\rho N(U-U^n,\ M-M^n). \tag{5.15}$$

Taking in to account the definition of $\gamma_{i,j}$ as scalar product in $L^2(\Gamma)$, with an easy calculation, we conclude that $\rho=\max$ [spectral radius of matrix $\{(|a_{i,j}|-h^2c_{i,j})/(a_{i,i}+h^2c_{i,i})\}$, spectral radius of $\{\gamma_{i,j}/\gamma_{i,i}\}]<1$ ([13], Th. 3.13) and thus

$$\lim N(U-U^n,\ M-M^n)=0\Rightarrow\lim U^n=U,\ \lim M^n=M. \tag{5.16}$$

§ 6. The Two Phase–Stefan Problem

Let

$]0, T[$ be a bounded interval of time variable t,

Ω an open bounded set in R^n with boundary $\Gamma=\Gamma_1\cup\Gamma_2$.

We consider a thermic phenomenon which involves a material in liquid and solid state. The temperature is expressed by $u(x, t)$, $(x, t)\in\Omega\times]0, T[$. The free boundary $L(t)$, whose equation is $t=l(x)$, an unknown of the problem at temperature zero, divides Ω into Ω_1 and Ω_2, where are located respectively the solid and liquid phases. Let

n be the normal unit vector to $L(t)$ directed towards Ω_1.

The specific heat c_i and the coefficient of the thermal conductivity k_i of body are known ($i=1, 2$) and have the properties:

$$c_i,\ k_i \text{ are positive constants defined in the interior } \mathring{\Omega}_i \text{ of } \Omega_i, \tag{6.1}$$

$$\beta_1=c_1/k_1>c_2/k_2=\beta_2>0. \tag{6.2}$$

We consider the following heat conduction problem ([3]):

$$\begin{cases} c_i\, \partial u/\partial t-k_i\cdot \Delta u=0 \quad \text{in } \mathring{\Omega}_i,\ i=1, 2, \\ u(x, t)\,|_{\Gamma_1}=\hat{b}(t), \\ \partial u/\partial n\,|_{\Gamma_2}=0; \end{cases} \tag{6.3}$$

$$\begin{cases} u\,|_{L(t)}=0, \\ k_1(\text{grad } u)_1\cdot n-k_2(\text{grad } u)_2\cdot n=LV_n \text{ on } L(t), \end{cases} \tag{6.4}$$

where V_n is the propagation velocity of $L(t)$ along n,

L is the latent heat of melting.

The initial condition is assigned:

$$u(x, 0)=u_0(x) \tag{6.5}$$

and we suppose $u_0(x)$ is strictly negative in Ω.

The author in [3] examines the circumstance of a degeneration of $L(t)$ in a mushy region, that is an interior non-empty region where $u(x, t)=0$ and by using the change of dependent variable:

$$v(x, t)=\int_0^t k_2u^+(x, \tau)-k_1u^-(x, \tau)\cdot d\tau, \tag{6.6}$$

he obtains the following weak setting of (6.3)—(6.5) in the form of variational inequality: to find $v(t)\in V$ such that

$$a(v, \omega-\partial v/\partial t)+\Phi(\omega)-\Phi(\partial v/\partial t)\geqslant c_1(u_0, \omega-\partial v/\partial t)_H, \quad \forall\omega\in V(t), \tag{6.7}$$

$$v(t)\,|_{\Gamma_1}=\int_0^t b(\tau)d\tau, \tag{6.8}$$

$$v(x, 0)=0, \tag{6.9}$$

where $V=H^1(\Omega)$,

$U(t)=\{\omega: \omega\in V/\omega|_{\Gamma_1}=b(t)\}$,

$H=L^2(\Omega)$,

$\Phi(\omega)=\int_\Omega \Phi_1(\omega)dx,\ \Phi_1(\omega)=1/2\beta_1(\omega^-)^2+1/2\beta_2(\omega^+)^2+L\omega^+$,

$b(t)=k_2(\hat{b}(t))^+-k_1(\hat{b}(t))^-$,

$a(v, \omega)=\int_\Omega \text{grad } v\times \text{grad } \omega\cdot dx.$

By introducing the proper convex function, lower semicontinuous in H ([11])

$$\chi(\omega)=\int_\Omega \omega^+\cdot dx \tag{6.10}$$

and setting

$$\beta\begin{cases}=\beta_1 & \text{if } \partial v/\partial t\leqslant 0, \\ =\beta_2 & \text{if } \partial v/\partial t\geqslant 0, \end{cases} \tag{6.11}$$

we can write (6.7) in the following form:

$$(\beta\partial v/\partial t,\ \omega-\partial v/\partial t)_H+a(v,\ \omega-\partial v/\partial t)$$
$$+L\chi(\omega)-L\chi(\partial v/\partial t)\geqslant c_1(u_0,\ \omega-\partial v/\partial t)_H. \tag{6.12}$$

We have

$$\partial(\omega^+)\begin{cases}=0 & \text{if } \omega<0,\\ =[0,1] & \text{if } \omega=0,\\ =1 & \text{if } \omega>0.\end{cases} \tag{6.13}$$

So we obtain, in the usual way, the following problem equivalent to (6.7)—(6.9): to find $\{v,\ \mu(\partial v/\partial t)\}\in V\times \Lambda$ such that

$$(\beta\partial v/\partial t,\ \omega)_H+a(v,\ \omega)+L(\mu(\partial v/\partial t),\ \omega)_H=c_1(u_0,\ \omega)_H,\quad \forall\omega\in V, \tag{6.14}$$
$$\omega|_{\Gamma_1}=0 \text{ and } (6.8),\ (6.9) \text{ satisfied},$$

where $\Lambda=\{\mu:\ \mu\in H/0\leqslant\mu\leqslant 1 \text{ a.e. on } \Omega\}$.

The problem (6.7)—(6.9), when $u_0\in L^2(\Omega)$, $b\in L^2(0,T)$, has a unique solution v such that $v,\ \partial v/\partial t\in L^2(0,T;V)$ ([3]). We shall use the formulation (6.14), (6.8), (6.9) to calculate the numerical approximation.

Discrete Problem

We suppose $n=2$; Ω a convex polygon and discretize (6.14) by the linear finite element method as regards space variables and by the finite difference method as regards time variable. We put $I_2=I_{\Gamma_1}\cup I_{\Gamma_2}$, where I_{Γ_i} is the index set related with nodes on Γ_i $(i=1,2)$; $I_2=I_0\cup I_{\Gamma_2}$.

We obtain the following implicit scheme: to find $\{v_h^m,\ \mu_h^m\}\in V_h\times\Lambda_h$ such that

$$\left(\beta\,\frac{v_h^m-v_h^{m-1}}{\delta},\ \omega_h\right)_H+a(v_h^m,\ \omega_h)+L\left(\mu\left(\frac{v_h^m-v_h^{m-1}}{\delta}\right),\ \omega_h\right)_H$$
$$=c_1(u_{0,h},\ \omega_h)_H,\quad \forall\omega_h\in V_h,\ \text{s.t. } \omega_h|_{\Gamma_1}=0,\ m=1,\ \cdots,\ M; \tag{6.15}$$

where $V_h=\{\phi_j^h\}_{I}$,

$\Lambda_h=\{\mu_h:\ \mu_h=\sum_{j\in I_2}M_j\phi_j^h/0\leqslant M_j\leqslant 1\}$,

$v_h^m=\sum_{j\in I}V_j^m\phi_j^h$ (V_j^m is calculated by integral (6.8) when $j\in I_{\Gamma_1}$),

$\delta=\Delta t=T/M,\quad M\in\mathbb{N}$,

$u_{0,h}$ is the interpolated function of u_0.

Algebraic System

$$\sum_{j\in I_2}\left[a_{i,j}Z_j^m+\frac{h^2}{\delta}c_{i,j}\beta_j^m Z_j^m+h^2Lc_{i,j}M_j^m\right]=\hat d_j,\quad i\in I_2,\ m=1,\ \cdots,\ M, \tag{6.16}$$

where $\beta_j^m\begin{cases}=\beta_1 & \text{if } Z_j^m\leqslant 0,\\ =\beta_2 & \text{if } Z_j^m>0;\end{cases}$

$d_i=c_1(u_{0,h},\ \phi_i^h)_H$,

$\hat d_i=d_i-\sum_{j\in I_2}a_{i,j}V_j^{m-1}-\sum_{j\in I_{\Gamma_1}}\left(a_{i,j}+\frac{h^2}{\delta}c_{i,j}\beta_j^m\right)Z_j^m$,

$M=\{M_i\}_{I_2}=\{M_i^m\}_{I_2}$,

$Z=Z^m=\{Z_i^m\}_{I_2}=\{V_i^m-V_i^{m-1}\}_{I_2}$.

If we set [illegible] system [illegible]

$$Z^{n+1}=\{Z_i^{m,n+1}\}_{I_2}=\{V_i^{m,n-1}-V_i^m\}_{I_2},$$

$$G_i(Z)=\hat{d}_i^m+\sum_{\substack{j\in I_2\\ j\neq i}}\left[\left(|a_{i,j}|-\frac{h^2}{\delta}c_{i,j}\beta_j^m\right)Z_j^m-h^2c_{i,j}M_j^m\right],$$

$$M^{n-1}=\{M_i^{m,n+1}\}_{I_2},$$

we can write (6.15) in the following form:

$$\left(a_{i,j}+\frac{h^2}{\delta}c_{i,i}\beta_i\right)Z_i+h^2Lc_{i,i}M_i=G_i(Z),\quad i\in I_2 \tag{6.17}$$

and we can calculate the solution $\{Z,\ M\}$ of (6.17) with an iterative Jacobi type method.

Algorithm

$$\begin{cases}\left(a_{i,i}+\dfrac{h^2}{\delta}c_{i,i}\beta_i\right)Z_i^{n+1}=G_i(Z^n), & \text{if } G_i(Z^n)<0,\\ h^2Lc_{i,i}M_i^{n+1}=G_i(Z^n), & \text{if } 0\leqslant G_i(Z^n)\leqslant h^2Lc_{i,i},\\ \left(a_{i,i}+\dfrac{h^2}{\delta}c_{i,i}\right)Z_i^{n+1}+h^2Lc_{i,i}=G_i(Z^n), & \text{if } G_i(Z^n)>h^2Lc_{i,i}.\end{cases} \tag{6.18}$$

Convergence of Algorithm

We obtain in this way a sequence $\{Z^n,\ M^n\}_n$ that converges to the solution $\{Z,\ M\}$ of (6.17); in fact starting from (6.17), (6.18) and keeping in mind the usual properties of matrices $A+h^2C$ and of subdifferential operator, we have the inequalities:

$$\begin{aligned}&\left(a_{i,i}+\frac{h^2}{\delta}c_{i,i}\hat{\beta}_i\right)|Z_i-Z_i^{n+1}|+h^2Lc_{i,i}|M_i-M_i^{n+1}|\\ &\quad\leqslant\sum_{\substack{j\in I_2\\ j\neq i}}[(|a_{i,j}|-hc_{i,j}\hat{\beta}_j)|Z_j-Z_j^n|+h^2Lc_{i,j}|M_j-M_j^n|],\end{aligned} \tag{6.19}$$

where $\hat{\beta}_j$ is a convenient choice from β_1, β_2. From (6.19) we obtain

$$N(Z-Z^{n+1},\ M-M^{n+1})\leqslant\rho(Z-Z^n,\ M-M^n), \tag{6.20}$$

where

$$N(Z-Z^n,\ M-M^n)=\operatorname*{usp}_{i\in I_2}(|Z_i-Z_i^n|+h\tau_i|M-M_i^n|),$$

$$\tau_i=Lc_{i,i}/\left(a_{i,i}+\frac{h^2}{\delta}c_{i,i}\hat{\beta}_i\right),$$

$$\begin{aligned}\rho=\max\ (&\text{spectral radius}\{(|a_{i,j}|-h^2c_{i,j}\hat{\beta}_j)/(a_{i,i}+h^2c_{i,i}\hat{\beta}_i)\},\\ &\text{spectral radius }\{c_{i,j}/c_{i,i}\})<1.\end{aligned}$$

We conclude that $\{Z_i^n,\ M^n\}\to\{Z,\ M\}$.

Remark I. The examples of Sections 2—6 suggest the belief that the explained method works well in more general situation, that is when the regularity properties of matrix A of discrete system (as (2.9)) permit to apply Jacobi, Gauss-Seidel and successive overrelaxation iterative methods ([13]), apart from the fact that the initial continuous problem is a minimum constrained problem, involving the Laplace operator. Clearly the presence of maximal-monotone term imposes, each time, some modifications to the proof of algorithm convergence of the same type introduced in Sections 2, 5, 6.

Remark II. The explained method can be used to solve quasi-variational inequalities and also the free plate unilateral problem ([4]) treated with the mixed finite element method.

§ 7. A Monotone Algorithm

To emphasize the character of the actual method and to prove its flexibility we consider now the "one obstacle" discrete problem (see Section 2):

$$\{U,\ M(U)\}\in R^{N_0}\times R^{N_0}:\ AU+M=b \text{ such that } U\geqslant A, \tag{7.1}$$

where $U=\{U_i\}_{I_0}$, $\mathbf{A}=\{A_i\}_{I_0}=\{\alpha(x_i)\}_{I_0}$,

$A=\{a_{i,j}\}_{I_0\times I_0}$ is a symmetric or non-symmetric matrix such that

i) A is positive definite,

ii) $a_{i,j}\leqslant 0,\ \forall i\neq j$.

Step 0 *of the Algorithm*

We consider the initial configuration $U^0=A$ and solve the system:

$$AU^0+M^0=b \tag{7.2}$$

in the unknown M^0.

If $M^0\leqslant 0$, $\{U^0,\ M^0\}$ is the solution of (7.1).

Step 1.

If $M^0\not\leqslant 0$ we consider the sets

$$Q=\{i:\ i\in I_0/M_i>0\},\quad Q'=I_0\backslash Q,$$

we put $M_Q=0$ and solve the subsystem:

$$A_{QQ}U_Q^1+A_{QQ'}U_{Q'}^0=b_Q \tag{7.3}$$

in the unknown U_Q^1.

As a consequence of monotonicity properties i), ii) of A ([13] p. 85 Corollary 2; p. 87 Ex. 7) we have $U^1=\{U_Q^1,\ U_{Q'}\}\geqslant U^0$. Then we solve the system

$$AU^1+M^1=b \tag{7.4}$$

in the unknown M^1.

If $M^1\leqslant 0$, $\{U^1,\ M^1\}$ is the solution of (7.1).

Step 2.

If $M^1\not\leqslant 0$, we consider the set $P=\{i:\ i\in I_0/M_i^1>0\}$ and put:

$$\text{new } Q=\text{old } Q\cup P,\quad M_Q=0$$

and solve the larger subsystem

$$A_{QQ}U_Q^2+A_{QQ'}U_{Q'}^1=b_Q \tag{7.5}$$

thus having $U^2=\{U_Q^2,\ U_{Q'}^1\}\geqslant U^1$. Then we solve the system

$$AU^2+M^2=b \tag{7.6}$$

in the unknown M^2.

If $M^2\leqslant 0$, $\{U^2,\ M^2\}$ is the solution of (7.1).

Continuing in this way we obtain the solution $\{U,\ M\}$ of (7.1) in a finite number of iterations.

For a full explanation of this type of algorithm without using monotone maximal operator we refer to [10], [12].

Remark III. In all algorithms 2—7 the iterative unknowns $\{U^n, M^n\}_n$ play the balanced, well defined roles of imposing, step by step, the respective constraints. Thus the peculiar feature of the method consists in working in primal and dual space at the same time. The presence of dual term facilitates the resolution of various problems and lends versatility to the method. In this respect the method carries out an original idea.

Remark IV. We want to announce in advance a profitable development of the method, concerning the elasto-plastic torsion of a bar (Section 4). The method enables us to express the dual term in the form:

$$\langle \mu_h, v_h \rangle = a(z_h, v_h).$$

In this way we obtain the equation:

$$\{u_h, z_h\} \in V_h \times V_h: a(u_h, v_h) + a(z_h, v_h) = \langle f, v_h \rangle, \quad \forall v_h \in V_h$$

under the constraint $|\mathrm{grad}\ u_h| \leqslant 1$.

We are using the successive overrelaxation iterative method to calculate the solution $\{u_h, z_h\}$. This algorithm turns out to be very efficient also in the unfavorable case of torsion forces depending on state variable x. The term z_h in this case has a physical meaning. The results of large numerical experiments will appear, in a short time in a special paper.

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