MATHEMATICAL ASPECT OF OPTIMAL CONTROL FINITE ELEMENT METHOD FOR NAVIER-STOKES PROBLEMS*

LI KAI-TAI (李开泰) HUANG AI-XIANG (黄艾香)
(Department of Mathematics, Xi'an Jiaotong University, Xi'an, China)

Abstract

This study deals with the theoretical basis of optimal control methods in primitive variable formulation and penalty function formulation of Navier-Stokes problems. Numerical examples demonstrating application are provided.

1. Introduction

The finite element formulation of the Navier-Stokes equation governing the flow of a viscous incompressible fluid can be classified into five basic categories: (1) primitive variable formulation (or velocity-pressure formulation), (2) penalty function formulation, (3) stream function formulation, (4) stream function-vorticity formulation, and (5) optimal control formulation. Each of them has relative advantages and disadvantages. These formulations differ mainly in the way the incompressibility condition is included in the formulation.

The optimal control formulation is to minimize the energy functional by the introduction of the state vector, which is a solution of a Stokes problem. The incompressibility condition is treated as a constraint or is eliminated to introduce the penalty function.

2. Optimal Control Formulation

We consider the following boundary value problem of the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\nabla)\boldsymbol{u}+\nabla p=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}|_{\Gamma_1}=0,\ \left(\nu\dfrac{\partial\boldsymbol{u}}{\partial n}-p\boldsymbol{n}\right)_{\Gamma_2}=\boldsymbol{g} & \text{on } \partial\Omega=\Gamma=\Gamma_1\cup\Gamma_2, \end{cases} \tag{2.1}$$

where $\boldsymbol{u}$ is the velocity of the fluid, p the pressure, Ω a bounded domain of R^n with a Lipschitz continuous boundary.

We introduce the Sobolev space $X=[H^1(\Omega)]^n$ with the norm

$$\|\boldsymbol{u}\|_1^2=\sum_{i=1}^n\|u_i\|_1^2, \quad \forall\boldsymbol{u}\in X,$$

* Received May 11, 1983.

and seminorm

$$|\boldsymbol{u}|_1^2=\sum_{i=1}^n |u_i|_1^2, \quad \forall \boldsymbol{u}\in X.$$

Let $X_0=[H_0^1(\Omega)]^n$, $V=\{\boldsymbol{u}|\boldsymbol{u}\in X,\ \boldsymbol{u}|_{\Gamma_1}=0\}$, $V_0=\{\boldsymbol{u}|\boldsymbol{u}\in V,\ \operatorname{div}\boldsymbol{u}=0\}$; so $X\subset V\subset X$ and $V_0\subset V$.

We also introduce linear, bilinear and trilinear functional:

$$a_0(\boldsymbol{u},\boldsymbol{v})=\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})=\nu\sum_{i,j=1}^n\int_\Omega \frac{\partial u^i}{\partial x^j}\frac{\partial v^i}{\partial x^j}dx,$$

$$G(\boldsymbol{u},\boldsymbol{v})=(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{v}),$$

$$a_\varepsilon(\boldsymbol{u},\boldsymbol{v})=a_0(\boldsymbol{u},\boldsymbol{v})+\varepsilon^{-1}G(\boldsymbol{u},\boldsymbol{v}), \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X. \tag{2.2}$$

$$a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=((\boldsymbol{u}\nabla)\boldsymbol{v},\boldsymbol{w})=\sum_{i,j=1}^n\int_\Omega u^j\frac{\partial v^i}{\partial x^j}w^i\,dx,$$

$$a(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})+a_1(\boldsymbol{v};\boldsymbol{u},\boldsymbol{w}),\ \langle\boldsymbol{F},\boldsymbol{v}\rangle=\langle\boldsymbol{f},\boldsymbol{v}\rangle+\langle\boldsymbol{g},\boldsymbol{v}\rangle_{\Gamma_2},$$

We assume $\boldsymbol{f}\in V'$ (dual space to V), $\boldsymbol{g}\in[H^{-1/2}(\Gamma_2)]^n$, and $H^{1/2}(\Gamma_2)=\{$the restriction to Γ_2 of $\gamma_0 g$, $g\in H^1(\Omega)\}$, γ_0 is the trace mapping from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$. If μ is in $H^{1/2}(\Gamma_2)$, we define

$$\|\mu\|_{1/2,\Gamma_2}=\inf_{g\in H^1(\Omega)}\{\|g\|_1,\ \mu=\gamma_0 g|_{\Gamma_2}\}.$$

Let $H^{-1/2}(\Gamma_2)^*$ be the dual space to $H^{1/2}(\Gamma_2)$, normed by

$$\|\mu^*\|_{-1/2,\Gamma_2}=\sup_{\mu\in H^{1/2}(\Gamma_2)}|\langle\mu^*,\mu\rangle_{\Gamma_2}|/\|\mu\|_{1/2,\Gamma_2},\quad \forall\mu^*\in H^{-1/2}(\Gamma_2),$$

where $\langle\cdot,\cdot\rangle_{\Gamma_2}$ denotes the duality between $H^{1/2}(\Gamma_2)$ and $H^{-1/2}(\Gamma_2)$. It is not difficult to prove $\langle\boldsymbol{f},\boldsymbol{v}\rangle+\langle\boldsymbol{g},\boldsymbol{v}\rangle_{\Gamma_2}$, $\forall\boldsymbol{f}\in V'$, $\boldsymbol{g}\in[H^{-1/2}(\Omega)]^n$, to be a linear continuous functional on the space V. Therefore there is an $\boldsymbol{F}\in V'$ such that

$$\langle\boldsymbol{F},\boldsymbol{v}\rangle=\langle\boldsymbol{f},\boldsymbol{v}\rangle+\langle\boldsymbol{g},\boldsymbol{v}\rangle_{\Gamma_2},\quad\forall\boldsymbol{v}\in V \tag{2.3}$$

and

$$\|\boldsymbol{F}\|_*\le\|\boldsymbol{f}\|_*+\|\boldsymbol{g}\|_{-1/2,\Gamma_2},$$

where $\|\cdot\|_*$ denotes the dual norm of V'.

In the velocity-pressure formulation, the variational form of (2.1) is

$$\begin{cases}\text{to find } \boldsymbol{u}\in V_0 \text{ such that}\\ a_0(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{F},\boldsymbol{v}\rangle,\quad\forall\boldsymbol{v}\in V_0.\end{cases} \tag{2.4}$$

In the penalty function formulation, the variational form of (2.1) is

$$\begin{cases}\text{to find } \boldsymbol{u}_\varepsilon\in V \text{ such that}\\ a_\varepsilon(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u}_\varepsilon;\boldsymbol{u}_\varepsilon,\boldsymbol{v})=\langle\boldsymbol{F},\boldsymbol{v}\rangle,\quad\forall\boldsymbol{v}\in V.\end{cases} \tag{2.5}$$

In both case, the variational form of (2.1) can be written as

$$\begin{cases}\text{to find } \boldsymbol{u}\in H \text{ such that}\\ A(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})=\langle\boldsymbol{F},\boldsymbol{v}\rangle,\quad\forall\boldsymbol{v}\in H.\end{cases} \tag{2.6}$$

In the case of (2.4), $H=V_0$ and $A(\boldsymbol{u},\boldsymbol{v})=a_0(\boldsymbol{u},\boldsymbol{v})$; in the case of (2.5), $H=V$ and $A(\boldsymbol{u},\boldsymbol{v})=a_\varepsilon(\boldsymbol{u},\boldsymbol{v})$.

We introduce the functional

$$J(\boldsymbol{v})=A(\boldsymbol{v}-\boldsymbol{\xi},\boldsymbol{v}-\boldsymbol{\xi})/2, \tag{2.7}$$

where $\boldsymbol{\xi}$ is a solution of the following Stokes problem:

$$\boldsymbol{\xi}\in H,\ A(\boldsymbol{\xi},\boldsymbol{\eta})=\langle\boldsymbol{F},\boldsymbol{\eta}\rangle-a_1(\boldsymbol{v};\boldsymbol{v},\boldsymbol{\eta}),\quad\forall\boldsymbol{\eta}\in H. \tag{2.8}$$

Of course ξ is a function of $\boldsymbol{v}$ through (2.8).

The optimal control problem associated with (2.6) is

$$\text{find } \boldsymbol{u}\in H \text{ such that } J(\boldsymbol{u})=\min_{\boldsymbol{v}\in H} J(\boldsymbol{v}). \tag{2.9}$$

It is obvious that (2.8), (2.9) have the structure of an optimal control problem where $\boldsymbol{v}$ is the control vector and ξ is the state vector. Equation (2.8) is a state equation, while the functional (2.7) is a cost function.

It is clear that if $\boldsymbol{u}$ satisfies (2.6), then $\boldsymbol{u}$ is also the solution of (2.7), (2.9). Inversely, if $\boldsymbol{u}$ is the solution of (2.8), (2.9) such that $J(\boldsymbol{u})=0$, then $\boldsymbol{u}$ also satisfies (2.6).

3. The State Variable

From now on assume $n\leqslant 4$. In this case the trilinear form $a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})$ is continuous on H. So we can introduce the norm of $a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})$ as follows:

$$N=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in H}\frac{|a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})|}{|\boldsymbol{u}|_1|\boldsymbol{v}|_1|\boldsymbol{w}|_1} \tag{3.1}$$

from which we have

$$|a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})|\leqslant N|\boldsymbol{u}|_1|\boldsymbol{v}|_1|\boldsymbol{w}|_1. \tag{3.2}$$

When meas $\Gamma_2=0$, $a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})$ is antisymmetric with respect to $\boldsymbol{v}$, $\boldsymbol{w}$:

$$a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=-a_1(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}),\quad \forall\boldsymbol{u}\in V_0,\ \boldsymbol{v},\boldsymbol{w}\in X \tag{3.3}$$
$$a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{v})=0.$$

(3.2) shows that $\forall\boldsymbol{u}\in H$, $\exists\boldsymbol{h}(\boldsymbol{u})\in H^1$ (dual space to H) such that

$$\langle\boldsymbol{h}(\boldsymbol{u}),\boldsymbol{v}\rangle=\langle\boldsymbol{F},\boldsymbol{v}\rangle-a_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in H \tag{3.4}$$

and

$$\|\boldsymbol{h}(\boldsymbol{u})\|_*=\sup_{\boldsymbol{v}\in H}|\langle\boldsymbol{h}(\boldsymbol{u}),\boldsymbol{v}\rangle|/|\boldsymbol{v}|_1\leqslant\|\boldsymbol{F}\|_*+N|\boldsymbol{u}|_1^2. \tag{3.5}$$

Especially, if $\boldsymbol{F}\in[L^{4/3}(\Omega)]^n$, then $\forall\boldsymbol{u}\in H$, $\boldsymbol{h}(\boldsymbol{u})\in[L^{4/3}(\Omega)]^n$,

$$\|\boldsymbol{h}(\boldsymbol{u})\|_{0,4/3}=\sup_{\boldsymbol{v}\in L^4(\Omega)}|\langle\boldsymbol{F},\boldsymbol{v}\rangle-a_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})|/\|\boldsymbol{v}\|_{0,4}\leqslant\|\boldsymbol{F}\|_{0,4/3}+C|\boldsymbol{u}|_1^2 \tag{3.6}$$

in view of the Sobolev embedding theorem $L^4(\Omega)\subsetneq H^1(\Omega)$.

Lemma 1. *The mapping defined by* (3.4) $\boldsymbol{u}\mapsto\boldsymbol{h}(\boldsymbol{u})$ *is a continuous operator from H into H^1. When $\boldsymbol{F}\in[L^{4/3}(\Omega)]^n$ it is also continuous from H into $[L^{4/3}(\Omega)]^n$. In the meantime the following estimates hold*

$$\|\boldsymbol{h}(\boldsymbol{u}_1)-\boldsymbol{h}(\boldsymbol{u}_2)\|_*\leqslant N(|\boldsymbol{u}_1|_1+|\boldsymbol{u}_2|_1)|\boldsymbol{u}_2-\boldsymbol{u}_1|_1,\quad\forall\boldsymbol{u}_1,\boldsymbol{u}_2\in H, \tag{3.7}$$

$$\|\boldsymbol{h}(\boldsymbol{u}_1)-\boldsymbol{h}(\boldsymbol{u}_2)\|_{0,4/3}\leqslant C(|\boldsymbol{u}_1|_1+|\boldsymbol{u}_2|_1)|\boldsymbol{u}_2-\boldsymbol{u}_1|_1,\quad\forall\boldsymbol{u}_1,\boldsymbol{u}_2\in H. \tag{3.8}$$

Proof. In fact,

$$\langle\boldsymbol{h}(\boldsymbol{u}_1)-\boldsymbol{h}(\boldsymbol{u}_2),\boldsymbol{v}\rangle=a_1(\boldsymbol{u}_2-\boldsymbol{u}_1;\boldsymbol{u}_2,\boldsymbol{v})+a_1(\boldsymbol{u}_1;\boldsymbol{u}_2-\boldsymbol{u}_1,\boldsymbol{v}),\qquad\forall\boldsymbol{v}\in H.$$

By virtue of (3.2), (3.5) and (3.6) and the Sobolev embedding inequality, (3.7) and (3.8) are obtained.

Bilinear functional $A(\boldsymbol{u},\boldsymbol{v})$ is continuous and coercive on $H\times H$:

$$\begin{aligned}&|A(\boldsymbol{u},\boldsymbol{v})|\leqslant M|\boldsymbol{u}|_1|\boldsymbol{v}|_1,\\&|A(\boldsymbol{u},\boldsymbol{u})|\geqslant\gamma|\boldsymbol{u}|_1^2.\end{aligned} \tag{3.9}$$

The introduction of (3.4) into (2.8) leads to

$$\text{find } \boldsymbol{\xi}\in H \text{ such that } A(\boldsymbol{\xi}, \boldsymbol{\eta}) = \langle \boldsymbol{h}(\boldsymbol{v}), \boldsymbol{\eta}\rangle, \quad \forall \boldsymbol{\eta}\in H. \tag{3.10}$$

Of course, there exists unique solution $\boldsymbol{\xi}=T\boldsymbol{v}$ for (3.10) according to Lax-Milgram's theorem, so the operator $\boldsymbol{v}\mapsto\boldsymbol{\xi}=T\boldsymbol{v}$ defined by (3.10) is the mapping from H into H, and

$$|T\boldsymbol{v}|_1\leqslant\nu^{-1}\|\boldsymbol{h}(\boldsymbol{v})\|_*\leqslant\nu^{-1}(\|\boldsymbol{F}\|_*+N|\boldsymbol{v}|_1^2), \quad \forall \boldsymbol{v}\in H. \tag{3.11}$$

In addition, when $\boldsymbol{F}\in[L^{4/3}(\Omega)]^n$ and meas $\Gamma_2=0$, then $T\boldsymbol{v}=\boldsymbol{\xi}\in[H^{2,4/3}(\Omega)]^n$ and

$$\|T\boldsymbol{v}\|_{2,4/3}\leqslant C\|\boldsymbol{h}(\boldsymbol{v})\|_{0,4/3}\leqslant C(\|\boldsymbol{F}\|_{0,4/3}+|\boldsymbol{v}|_1^2), \quad \forall \boldsymbol{v}\in H. \tag{3.12}$$

In other words, T is a mapping from H into $[H^{2,4/3}(\Omega)]^n$. In this case, using (3.7), (3.8) and Lax-Milgram's theorem we can obtain:

Lemma 2. *Suppose Ω is a bounded domain of R^n with Lipschitz boundary Γ. Then the mapping T defined by* (3.10) *is continuous from H into H. If the boundary is of class C^2, $\boldsymbol{F}\in[L^{4/3}(\Omega)]^n$, and meas $\Gamma_2=0$, then T is also continuous from H into $[H^{2,4/3}(\Omega)]^n$ and the following holds:*

$$|T\boldsymbol{v}_1-T\boldsymbol{v}_2|_1\leqslant N\nu^{-1}(|\boldsymbol{v}_1|_1+|\boldsymbol{v}_2|_1)|\boldsymbol{v}_1-\boldsymbol{v}_2|_1, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2\in H, \tag{3.13}$$

$$\|T\boldsymbol{v}_1-T\boldsymbol{v}_2\|_{2,4/3}\leqslant C(|\boldsymbol{v}_1|_1+|\boldsymbol{v}_2|_1)|\boldsymbol{v}_1-\boldsymbol{v}_2|_1, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2\in H. \tag{3.14}$$

From Lemmas 1 and 2 we immediately obtain:

Lemma 3. *Suppose Ω is a bounded domain of R^n with boundary of class C^2, $\boldsymbol{F}\in[L^{4/3}(\Omega)]^n$ and meas $\Gamma_2=0$. Then the operator T from H into H, defined by* (3.10), *is compact.*

Lemma 4. *Suppose Ω is a bounded domain of R^n with Lipschitz boundary $\Gamma=\Gamma_1\cup\Gamma_2$, $H=V_0$. Then the form $\boldsymbol{u}\mapsto a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})$ is weakly continuous in H, i. e.*

$$\boldsymbol{u}_m\to\boldsymbol{u}\,(\textit{weakly}) \textit{ in } H\Rightarrow\lim_{m\to\infty}a_1(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v})=a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{v}\in H. \tag{3.15}$$

Proof. Since the embedding operator $H\subsetneq[L^2(\Omega)]^n$ is compact and an arbitrary linear compact operator from a reflexive Banach space into a Banach space is certainly a strongly continuous operator, so $\boldsymbol{u}_m\to\boldsymbol{u}$ (strongly) in $[L^2(\Omega)]^n$.

Now, $\forall \boldsymbol{u}\in H$, $\forall \boldsymbol{u}, \boldsymbol{w}\in X$,

$$a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w})+a_1(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})=\oint_\Gamma(\boldsymbol{v}\boldsymbol{w})(\boldsymbol{u}\boldsymbol{n})ds-\int_\Omega \boldsymbol{v}\boldsymbol{w}\operatorname{div}\boldsymbol{u}\,dx=\int_{\Gamma_2}(\boldsymbol{v}\boldsymbol{w})(\boldsymbol{u}\boldsymbol{n})ds.$$

Hence

$$a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w})=-a(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v})+\int_{\Gamma_2}(\boldsymbol{v}\boldsymbol{w})(\boldsymbol{u}\boldsymbol{n})ds.$$

Let $\boldsymbol{w}\in[\mathscr{D}(\Omega)]^n$; then

$$\begin{aligned}&|a_1(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{w})-a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w})|\\&=\Big|a_1(\boldsymbol{u}-\boldsymbol{u}_m; \boldsymbol{w}, \boldsymbol{u})+a_1(\boldsymbol{u}_m; \boldsymbol{w}, \boldsymbol{u}-\boldsymbol{u}_m)\\&\quad+\int_{\Gamma_2}(\boldsymbol{u}_m-\boldsymbol{u})\boldsymbol{w}(\boldsymbol{u}_m\boldsymbol{n})\,ds-\int_{\Gamma_2}(\boldsymbol{u}\boldsymbol{w})(\boldsymbol{u}_m-\boldsymbol{u})\boldsymbol{n}\,ds\Big|\\&\leqslant N\|\boldsymbol{u}-\boldsymbol{u}_m\|_{0,2}|\boldsymbol{w}|_{1,4}(\|\boldsymbol{u}\|_{0,4}+\|\boldsymbol{u}_m\|_{0,4})\\&\quad+C\|\boldsymbol{u}_m-\boldsymbol{u}\|_{0,8/3,\Gamma}|\boldsymbol{w}|_{0,8/3,\Gamma}(|\boldsymbol{u}|_{0,4,\Gamma}+|\boldsymbol{u}_m|_{0,4,\Gamma}).\end{aligned}$$

It is well known that as $q<2(n-1)/(n-2)$, $H^1(\Omega)\subsetneq L^q(\Gamma)$ is compact, and $H^1(\Omega)\subsetneq L^4(\Omega)$. From this we have $\boldsymbol{u}_m\to\boldsymbol{u}$(strongly) in $[L^{8/3}(\Gamma)]^n$ and

$$|a_1(\boldsymbol{u}_m;\ \boldsymbol{u}_m,\ \boldsymbol{w})-a_1(\boldsymbol{u};\ \boldsymbol{u},\ \boldsymbol{w})|$$
$$\leqslant CN|\boldsymbol{w}|_{1,4}(|\boldsymbol{u}|_1+|\boldsymbol{u}_m|_1)\|\boldsymbol{u}-\boldsymbol{u}_m\|_{0,2}+C|\boldsymbol{w}|_1(|\boldsymbol{u}|_1+|\boldsymbol{u}_m|_1)\|\boldsymbol{u}_m-\boldsymbol{u}\|_{0,8/3,\Gamma}.$$

Therefore

$$\lim_{m\to\infty} a_1(\boldsymbol{u}_m;\ \boldsymbol{u}_m,\ \boldsymbol{w})=a_1(\boldsymbol{u};\ \boldsymbol{u},\ \boldsymbol{w})$$

in view of the boundedness of $\{\boldsymbol{u}_n\}$ in H. Thus (3.15) holds by virtue of the density of $[\mathscr{D}(\Omega)]^n$ in H and the trilinearity of $a_1(\cdot;\ \cdot,\ \cdot)$.

It is easy to obtain the following lemma from Lemma 4.

Lemma 5. *Suppose the condition of Lemma 4 is satisfied. Then the operator T defined by (3.10) is weakly continuous from H into H:*

$$\boldsymbol{u}_m\to\boldsymbol{u}\,(weakly)\ in\ H\Rightarrow T\boldsymbol{u}_m\to T\boldsymbol{u}\,(weakly)\ in\ H. \tag{3.16}$$

Lemma 6. *Under the conditions of Lemmas 3 and 4 and $\boldsymbol{F}\in[L^{4/3}(\Omega)]^n$, the operator T defined by (3.10) is strongly continuous from H into H:*

$$\boldsymbol{u}_m\to\boldsymbol{u}\,(weakly)\ in\ H\Rightarrow T\boldsymbol{u}_m\to T\boldsymbol{u}\,(strongly)\ in\ H. \tag{3.17}$$

Proof. Suppose (3.17) is not true. Then there exists a subsequence $\{\boldsymbol{u}_{mp}\}$ such that

$$|T\boldsymbol{u}_{mp}-T\boldsymbol{u}|_1>\varepsilon,\quad \forall\varepsilon>0. \tag{3.18}$$

But we also have $\boldsymbol{u}_{mp}\to\boldsymbol{u}$ in H and hence $\{\boldsymbol{u}_{mp}\}$ is uniformly bounded in H. As T is compact, there exists a subsequence $\{\boldsymbol{u}_{mq}\}$ of $\{\boldsymbol{u}_{mp}\}$ with $T\boldsymbol{u}_{mq}\to\boldsymbol{w}$ in H. On the other hand, $\boldsymbol{u}_{mq}\to\boldsymbol{u}\Rightarrow T\boldsymbol{u}_{mq}\to T\boldsymbol{u}$ by virtue of Lemma 5. As the weak limit is unique we conclude that $\boldsymbol{w}=T\boldsymbol{u}$, and thus $T\boldsymbol{u}_{mq}\to T\boldsymbol{u}$ in H. This contradicts (3.18), and so we conclude that $T\boldsymbol{u}_m\to T\boldsymbol{u}$ in H, i.e. T is strongly continuous.

Lemma 7. *The operator T is Gateaux differentiable everywhere in H:*

$$A(T'(\boldsymbol{u})\boldsymbol{w},\ \boldsymbol{v})=-a_1(\boldsymbol{u};\ \boldsymbol{w},\ \boldsymbol{v})-a_1(\boldsymbol{w};\ \boldsymbol{u},\ \boldsymbol{v}),\quad \forall\boldsymbol{v},\ \boldsymbol{w}\in H. \tag{3.19}$$

and $T'(u)$ is Lipschitz continuous in H:

$$\|T'(\boldsymbol{u}_1)-T'(\boldsymbol{u}_2)\|\leqslant 2N\nu^{-1}|\boldsymbol{u}_1-\boldsymbol{u}_2|_1,\quad \forall\boldsymbol{u}_1,\ \boldsymbol{u}_2\in H. \tag{3.20}$$

Furthermore, if the conditions of Lemma 3 are satisfied, $\forall\boldsymbol{u}\in H$, $T'(\boldsymbol{u})$ is compact for $H\to H$.

Proof. It is not difficult to obtain (3.19). To prove (3.20) we observe that

$$A((T'(\boldsymbol{u}_1)-T'(\boldsymbol{u}_2))\boldsymbol{w},\ \boldsymbol{v})=a_1(\boldsymbol{u}_2-\boldsymbol{u}_1;\ \boldsymbol{w},\ \boldsymbol{v})+a_1(\boldsymbol{w};\ \boldsymbol{u}_2-\boldsymbol{u}_1,\ \boldsymbol{v}).$$

Hence

$$\|T'(\boldsymbol{u}_1)-T'(\boldsymbol{u}_2)\|\leqslant\nu^{-1}\sup_{\boldsymbol{v},\boldsymbol{w}\in H}\frac{|a_1(\boldsymbol{u}_2-\boldsymbol{u}_1;\ \boldsymbol{w},\ \boldsymbol{v})+a_1(\boldsymbol{w};\ \boldsymbol{u}_2-\boldsymbol{u}_1,\ \boldsymbol{v})|}{|\boldsymbol{w}|_1|\boldsymbol{v}|_1}.$$

Employing (3.2) we obtain (3.20).

By virtue of (3.6) and the regular theorem for Stokes problems, $T'(\boldsymbol{u})\boldsymbol{w}\in H^{2,4/3}(\Omega)\subsetneq H$. Hence $T'(u)$ is compact.

Let $t\in R^1$. It follows that

$$A(T(\boldsymbol{u}+t\boldsymbol{w}),\ \boldsymbol{v})=\langle\boldsymbol{F},\ \boldsymbol{v}\rangle-a_1(\boldsymbol{u}+t\boldsymbol{w};\ \boldsymbol{u}+t\boldsymbol{w},\ \boldsymbol{v}).$$

By virtue of (3.19) we obtain

$$T(\boldsymbol{u}+t\boldsymbol{w})=T\boldsymbol{u}+T'(\boldsymbol{u})\boldsymbol{w}t+T'(\boldsymbol{w})\boldsymbol{w}t^2/2. \tag{3.21}$$

4. The Existence Theorem

The variational problem (2.8) is equivalent to the following operator equation

$$\boldsymbol{u}=T\boldsymbol{u}. \tag{4.1}$$

In other words, $\boldsymbol{u}$ is a solution of (2.8) if and only if $\boldsymbol{u}$ is a fixed point of T.

Theorem 1. *Suppose $\boldsymbol{F}\in H^1$ and*

$$4N\nu^{-2}\|\boldsymbol{F}\|_*<1. \tag{4.2}$$

Let

$$K=\{\boldsymbol{u}\,|\,\boldsymbol{u}\in H,\ |\boldsymbol{u}|_1\leqslant 2\nu^{-1}\|\boldsymbol{F}\|_*\}. \tag{4.3}$$

Then there exists a unique fixed point of T in the set K.

Proof. First, we prove that T is a mapping $K\to K$. If $\boldsymbol{v}\in K$, then

$$|T\boldsymbol{v}|_1\leqslant\nu^{-1}(\|\boldsymbol{F}\|_*+N|\boldsymbol{v}|_1^2)\leqslant\nu^{-1}(\|\boldsymbol{F}\|_*+4N\nu^{-2}\|\boldsymbol{F}\|_*^2)\leqslant 2\nu^{-1}\|\boldsymbol{F}\|_*$$

by virtue of (3.11) and (4.2). So T: $K\to K$.

Secondly, T is a contraction mapping in K. In fact, if $\boldsymbol{v}_1,\ \boldsymbol{v}_2\in K$, then in view of (4.2),

$$|T\boldsymbol{v}_1-T\boldsymbol{v}_2|_1\leqslant N\nu^{-1}(|\boldsymbol{v}_1|_1+|\boldsymbol{v}_2|_1)\,|\boldsymbol{v}_1-\boldsymbol{v}_2|_1<|\boldsymbol{v}_1-\boldsymbol{v}_2|_1.$$

So T is a contraction mapping $K\to K$. We conclude that the mapping T has a unique fixed point in K.

It is well known that if $A(u,\ v)=a_0(\boldsymbol{u},\ \boldsymbol{v})$, $H=V_0$ and meas $\Gamma_2=0$, then problem (4.1) has at least a solution $\boldsymbol{u}^*$ which satisfies

$$|\boldsymbol{u}^*|_1\leqslant\nu^{-1}\|\boldsymbol{F}\|_*. \tag{4.4}$$

In addition, if

$$N\nu^{-2}\|\boldsymbol{F}\|_*<1, \tag{4.4*}$$

then problem (4.1) has a unique solution which satisfies (4.4). In another case we have:

Theorem 2. *Suppose $A(\boldsymbol{u},\ \boldsymbol{v})=a_s(\boldsymbol{u},\ \boldsymbol{v})$, $\boldsymbol{F}\in H^1$, $H=V$, meas $\Gamma_2=0$, and the constants α, β are such that*

$$\nu>\alpha>0,\quad \nu-c\beta/2\geqslant\alpha>0$$

where c is a Sobolev embedding constant

$$\|\boldsymbol{u}\|_{0,4}\leqslant c|\boldsymbol{u}|_1,\quad \forall\boldsymbol{u}\in X,\ n\geqslant 4.$$

Then the penalty variational problem (2.5) for the Navier-Stokes equation has a unique solution in the set K_1:

$$K_1=\{\boldsymbol{u}\,|\,\boldsymbol{u}\in V,\ \|\mathrm{div}\,\boldsymbol{u}\|_0\leqslant\beta,\ |\boldsymbol{u}|_1\leqslant\alpha^{-1}\|\boldsymbol{F}\|_*\} \tag{4.5}$$

if

(i)
$$\alpha^{-1}N\|\boldsymbol{F}\|_*<1, \tag{4.6}$$

(ii) *the penalty parameter ε is small enough such that*

$$0.5(\varepsilon/\alpha)^{1/2}\|\boldsymbol{F}\|_*\leqslant\beta. \tag{4.7}$$

Proof. It is easy to prove that

$$|a_1(\boldsymbol{u};\ \boldsymbol{v},\ \boldsymbol{v})|\leqslant 0.5c|\boldsymbol{v}|_1^2\|\mathrm{div}\,\boldsymbol{u}\|_0,\quad \forall\boldsymbol{u},\ \boldsymbol{v}\in H. \tag{4.8}$$

Let $C_u(\boldsymbol{w},\ \boldsymbol{v})=a_s(\boldsymbol{w},\ \boldsymbol{v})+a_1(\boldsymbol{u};\ \boldsymbol{w},\ \boldsymbol{v})$. Then $\forall\boldsymbol{u}\in K_1$,

(1) $C_u(\cdot, \cdot)$ is a continuous bilinear form on $H\times H$,

(2) $C_u(\cdot, \cdot)$ is H-elliptic:

$$C_u(\boldsymbol{v}, \boldsymbol{v})\geqslant\alpha|\boldsymbol{v}|_1^2, \quad \forall\boldsymbol{v}\in H. \tag{4.9}$$

In fact, by (4.8), (4.5) and (4.4), $\forall\boldsymbol{u}\in K_1$,

$$C_u(\boldsymbol{v}, \boldsymbol{v})=\nu|\boldsymbol{v}|_1^2+\varepsilon^{-1}\|\operatorname{div}\boldsymbol{v}\|_0^2+a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{v})\geqslant(\nu-c\beta/2)|\boldsymbol{v}|_1^2\geqslant\alpha|\boldsymbol{v}|_1^2, \quad \forall\boldsymbol{v}\in H.$$

Hence the existence of the unique solution to

$$C_u(\boldsymbol{w}, \boldsymbol{v})=\langle\boldsymbol{F}, \boldsymbol{v}\rangle, \quad \forall\boldsymbol{v}\in H \tag{4.10}$$

is guaranteed by Lax-Milgram's theorem. That is, a mapping $\boldsymbol{w}=P\boldsymbol{u}$ is well defined by (4.10)

$$|P\boldsymbol{u}|_1=|\boldsymbol{w}|_1\leqslant\alpha^{-1}\|\boldsymbol{F}\|_*.$$

In addition, by virtue of (4.10) and (4.9) we have

$$\varepsilon^{-1}\|\operatorname{div}\boldsymbol{w}\|_0^2+\alpha|\boldsymbol{w}|_1^2\leqslant\|\boldsymbol{F}\|_*|\boldsymbol{w}|_1.$$

Using $ab\leqslant\sigma a^2+b^2/(4\sigma)$ (where $\sigma>0$ is an arbitrary constant) we have

$$\varepsilon^{-1}\|\operatorname{div}\boldsymbol{w}\|_0^2+(\alpha-\sigma)|\boldsymbol{w}|_1^2\leqslant\|\boldsymbol{F}\|_*^2/(4\sigma).$$

Setting $\alpha=\sigma$ and using (4.7) we obtain

$$\|\operatorname{div}\boldsymbol{w}\|_0\leqslant\beta.$$

Consequently P is the mapping $K_1\to K_1$.

Furthermore, P is also a contraction mapping. In fact, if $\boldsymbol{w}_1=P\boldsymbol{u}_1$ and $\boldsymbol{w}_2=P\boldsymbol{u}_2$, we have

$$a_\varepsilon(\boldsymbol{w}_1-\boldsymbol{w}_2, \boldsymbol{v})+a_1(\boldsymbol{u}_1-\boldsymbol{u}_2; \boldsymbol{w}_1, \boldsymbol{v})+a_1(\boldsymbol{u}_2; \boldsymbol{w}_1-\boldsymbol{w}_2, \boldsymbol{v})=0.$$

Setting $\boldsymbol{v}=\boldsymbol{w}_1-\boldsymbol{w}_2$, we obtain

$$C_{u_2}(\boldsymbol{w}_1-\boldsymbol{w}_2, \boldsymbol{w}_1-\boldsymbol{w}_2)=a_1(\boldsymbol{u}_2-\boldsymbol{u}_1; \boldsymbol{w}_1, \boldsymbol{w}_1-\boldsymbol{w}_2).$$

It follows that

$$|\boldsymbol{w}_1-\boldsymbol{w}_2|_1\leqslant N\alpha^{-2}\|\boldsymbol{F}\|_*|\boldsymbol{u}_1-\boldsymbol{u}_2|_1$$

by virtue of $\boldsymbol{w}_1\in K_1$. Using (4.6) we conclude that P is a contraction mapping. So $\boldsymbol{w}=P\boldsymbol{w}$ has a unique fixed point in K_1.

Assume the $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ is the solution of (2.5) and $(\boldsymbol{u}, p)$ is the solution of (2.4). Now we can state the following results.

Theorem 3. *Under hypothesis* (4.2) *the following estimate holds*

$$|\boldsymbol{u}-\boldsymbol{u}_\varepsilon|_1+\|p-p_\varepsilon\|_0\leqslant c_1\varepsilon,$$

where c_1 is a constant.

For the proof see [13].

Lemma 8. *Assume $\boldsymbol{u}^*$ is a solution of* (4.1) *and condition* (4.2) *is satisfied. Then 1 is not an eigenvalue of $T'(\boldsymbol{u}^*)$, i.e.*

$$\boldsymbol{w}-T'(\boldsymbol{u}^*)\boldsymbol{w}=0\Rightarrow\boldsymbol{w}=0. \tag{4.11}$$

Proof. In fact,

$$A(\boldsymbol{w}-T'(\boldsymbol{u}^*)\boldsymbol{w}, \boldsymbol{v})=A(\boldsymbol{w}, \boldsymbol{v})+a_1(\boldsymbol{u}^*; \boldsymbol{w}, \boldsymbol{v})+a_1(\boldsymbol{w}; \boldsymbol{u}^*, \boldsymbol{v}), \quad \forall\boldsymbol{v}\in H.$$

Let $C(\boldsymbol{u}^*; \boldsymbol{w}, \boldsymbol{v})=A(\boldsymbol{w}-T'(\boldsymbol{u}^*)\boldsymbol{w}, \boldsymbol{v})$. Employing (3.9), (3.2), (4.2) and (4.3) we obtain

$$C(\boldsymbol{u}^*; \boldsymbol{w}, \boldsymbol{w})\geqslant\nu(1-4N\nu^{-2}\|\boldsymbol{F}\|_*)|\boldsymbol{w}|_1^2>|\boldsymbol{w}|_1^2, \quad \forall\boldsymbol{w}\in H. \tag{4.12}$$

That is coerciveness of $C(\boldsymbol{u}^*; \cdot, \cdot)$. On the other hand, $C(\boldsymbol{u}^*; \cdot, \cdot)$ is also continuous on $H\times H$. Hence

$$C(\boldsymbol{u}^*;\ \boldsymbol{w},\ \boldsymbol{v})=0,\qquad \forall \boldsymbol{v}\in H\Rightarrow \boldsymbol{w}=0,$$

i.e. (4.11) is obtained.

5. The Minimizing Sequence

Let $J(\boldsymbol{u})$ be a functional defined by (2.8)

$$J(\boldsymbol{u})=A(\boldsymbol{u}-T\boldsymbol{u},\ \boldsymbol{u}-T\boldsymbol{u})=A(\boldsymbol{u}-\boldsymbol{\xi},\ \boldsymbol{u}-\boldsymbol{\xi}).\tag{5.1}$$

It is easy to check that $J(\boldsymbol{u})$ is Gateaux differentiable everywhere in H, and

$$\begin{aligned}\langle \text{Grad } J(\boldsymbol{u}),\ \boldsymbol{w}\rangle &= A(T\boldsymbol{u}-\boldsymbol{u},\ \boldsymbol{w}-T'(\boldsymbol{u})\boldsymbol{w})\\ &= A(T\boldsymbol{u}-\boldsymbol{u},\ \boldsymbol{w})+a_1(\boldsymbol{u};\ \boldsymbol{w},\ T\boldsymbol{u}-\boldsymbol{u})\\ &\quad +a_1(\boldsymbol{w}:\ \boldsymbol{u},\ T\boldsymbol{u}-\boldsymbol{u}),\qquad \forall \boldsymbol{u},\ \boldsymbol{w}\in H.\end{aligned}\tag{5.2}$$

From this we conclude that if $\boldsymbol{u}^*$ is a solution of (4.1), i.e. a fixed point of T, then $\boldsymbol{u}^*$ is a stationary point of J:

$$\text{Grad } J(\boldsymbol{u}^*)=0.\tag{5.3}$$

If $\boldsymbol{u}^*$ is a stationary point of J, then we have, by (5.2),

$$A(T\boldsymbol{u}^*-\boldsymbol{u}^*,\ \boldsymbol{w}-T'(\boldsymbol{u}^*)\boldsymbol{w})=0,\quad \forall \boldsymbol{w}\in H.\tag{5.4}$$

From Lemmas 7 and 8 we conclude that $I-T'(\boldsymbol{u}^*)$ is an isomorphism of H. So (5.4) yields

$$T\boldsymbol{u}^*-\boldsymbol{u}^*=0.$$

Hence we obtain the following theorem:

Theorem 4. *Under condition (4.2), a solution of (4.1) is a stationary point of J. Inversely, a stationary point of J is also a solution of (4.1).*

Theorem 5. *Suppose Ω is a bounded domain of R^n with Lipschitz boundary Γ and each of the following two hypotheses holds:*

(1) $$A(\cdot,\ \cdot)=a_0(\cdot,\ \cdot),\quad H=V_0,\quad \boldsymbol{F}\in V_0',\tag{5.5}$$

(2) $$A(\cdot,\ \cdot)=a_s(\cdot,\ \cdot),\quad H=X_0,\quad \boldsymbol{F}\in[L^{4/3}(\Omega)]^n,\ \Gamma \text{ is of class } C^2.\tag{5.6}$$

Then $J(\boldsymbol{u})$ is weakly lower semicontinuous on H and achieves its infimum at some point in H.

Proof. Let $\boldsymbol{u}_m\to \boldsymbol{u}$ (weakly) in H.

(1) When (5.5) is valid, then by Lemma 5,

$$T\boldsymbol{u}_m\to T\boldsymbol{u}\text{(weakly)}\quad \text{in } H.$$

(2) When (5.6) is valid, then there exists a subsequence (still denoted by $\{T\boldsymbol{u}_m\}$) such that, by Lemma 3,

$$T\boldsymbol{u}_m\to T\boldsymbol{u}\ \text{(strongly)}\quad \text{in } H.$$

So we have $\boldsymbol{z}_m=\boldsymbol{u}_m-T\boldsymbol{u}_m\to \boldsymbol{z}=\boldsymbol{u}-T\boldsymbol{u}$ (weakly) in H in either case. In addition,

$$A(\boldsymbol{z}_m-\boldsymbol{z},\ \boldsymbol{z}_m-\boldsymbol{z})\geqslant 0\Rightarrow A(\boldsymbol{z}_m,\ \boldsymbol{z}_m)\geqslant 2A(\boldsymbol{z}_m,\ \boldsymbol{z})-A(\boldsymbol{z},\ \boldsymbol{z}).$$

Hence we get

$$\liminf J(\boldsymbol{u}_m)\geqslant J(\boldsymbol{u}).\tag{5.7}$$

Similarly we can prove that (5.7) is also true for the whole sequence $\{\boldsymbol{u}_m\}$. So we conclude that J is weakly lower semicontinuous on H.

On the other hand, $J(\boldsymbol{u})\geqslant 0,\ \forall \boldsymbol{u}\in H$. Let $\alpha=\inf\limits_{\boldsymbol{u}\in H} J(\boldsymbol{u})$. Assume $\{\boldsymbol{u}_m\}$ is a minimizing sequence

$$\lim_{m\to\infty} J(\boldsymbol{u}_m)=\alpha.$$

From this,

$$A(\boldsymbol{z}_m,\ \boldsymbol{z}_m)\leqslant M.$$

Hence $|\boldsymbol{z}_m|_1\leqslant c$ by virtue of (3.9), i. e. $\{\boldsymbol{z}_m\}$ is uniformly bounded. Furthermore, it follows from equation (2.8) that

$$A(\boldsymbol{u}_m,\ \boldsymbol{v})+a_1(\boldsymbol{u}_m;\ \boldsymbol{u}_m,\ \boldsymbol{v})=\langle \boldsymbol{F},\ \boldsymbol{v}\rangle-A(\boldsymbol{z}_m,\ \boldsymbol{v}). \tag{5.8}$$

So

$$|\boldsymbol{u}_m|_1\leqslant \nu^{-1}(\|\boldsymbol{F}\|_*+\nu|\boldsymbol{z}_m|_1)\leqslant C_2.$$

In the case of (5.6), setting $\boldsymbol{v}=\boldsymbol{u}_m$ in (5.8)

$$(\nu-0.5c\|\operatorname{div}\ \boldsymbol{u}_m\|_0)\,|\boldsymbol{u}_m|_1\leqslant\|\boldsymbol{F}\|_*+\nu|\boldsymbol{z}_m|_1\leqslant\nu C_2$$

by virtue of (4.8). From the proof of Theorem 2 we know that if s is small enough we have $\nu-0.5c\|\operatorname{div}\ \boldsymbol{u}_m\|_0\geqslant\nu-0.5c\beta\geqslant\alpha$. So $\{\boldsymbol{u}_m\}$ is uniformly bounded. Therefore we can extract a subsequence $\{\boldsymbol{u}_{mp}\}$ of $\{\boldsymbol{u}_m\}$ such that

$$\boldsymbol{u}_{mp}\to\boldsymbol{u}_0(\text{weakly})\quad\text{in } H$$

and

$$\lim_{mp\to\infty} J(\boldsymbol{u}_{mp})=\inf_{\boldsymbol{v}\in H} J(\boldsymbol{v})=\alpha. \tag{5.9}$$

Because $J(\boldsymbol{u})$ is weakly lower semicontinuous on H we have

$$\alpha=\lim_{mp\to\infty} J(\boldsymbol{u}_m)\geqslant J(\boldsymbol{u}_0). \tag{5.10}$$

But by the definition of α we must have $J(\boldsymbol{u}_0)\geqslant\alpha$. This shows $J(\boldsymbol{u}_0)=\alpha$.

Theorem 6. *Suppose the conditions of Lemma 2 are satisfied. The minimizing sequence $\{\boldsymbol{u}_m\}$ of J is such that*

$$\lim_{m\to\infty} J(\boldsymbol{u}_m)=0. \tag{5.11}$$

Then $\{\boldsymbol{u}_m\}$ converges strongly to the solution $\boldsymbol{u}$ of (2.6):

$$\boldsymbol{u}_m\to\boldsymbol{u}_0\quad in\ H. \tag{5.12}$$

Proof. In the proof of Theorem 4 we showed that there exists a subsequence $\boldsymbol{u}_{mp}$ of $\boldsymbol{u}_m$ such that $\boldsymbol{u}_{mp}\to\boldsymbol{u}_0$, $J(\boldsymbol{u}_0)=\inf\ J(\boldsymbol{v})=0$. According to Lemma 3 there also exists a subsequence (still denoted by $\boldsymbol{u}_{mp}$) of $\boldsymbol{u}_{mp}$ such that

$$T\boldsymbol{u}_{mp}\to T\boldsymbol{u}_0\ (\text{strongly})\quad\text{in } H. \tag{5.13}$$

In addition, (5.11) shows that

$$\lim_{m\to\infty} A(\boldsymbol{z}_m,\ \boldsymbol{z}_m)=0,\qquad \boldsymbol{z}_m=\boldsymbol{u}_m-T\boldsymbol{u}_m,$$

i. e.

$$\lim_{m\to\infty}|\boldsymbol{z}_m|_1=0. \tag{5.14}$$

From (5.13) and (5.14) we conclude that

$$\boldsymbol{u}_{mp}\to\boldsymbol{u}_0(\text{strongly})\quad\text{in } H. \tag{5.15}$$

Using (5.5) we have

$$A(\boldsymbol{u}_0,\ \boldsymbol{v})+a_1(\boldsymbol{u}_{mp};\ \boldsymbol{u}_{mp},\ \boldsymbol{v})=\langle\boldsymbol{F},\ \boldsymbol{v}\rangle-A(\boldsymbol{z}_{mp},\ \boldsymbol{v}),\quad\forall\boldsymbol{v}\in H.$$

Letting $mp\to+\infty$ we get

$$A(\boldsymbol{u}_0,\ \boldsymbol{v})+a_1(\boldsymbol{u}_0;\ \boldsymbol{u}_0,\ \boldsymbol{v})=\langle\boldsymbol{F},\ \boldsymbol{v}\rangle,\quad\forall\boldsymbol{v}\in H \tag{5.16}$$

by virtue of (5.14) and (5.15), i.e. $\boldsymbol{u}_0$ is a solution of (2.6).

It remains to prove (5.12). Assume that it is not true. Then there exists a subsequence $\boldsymbol{u}_{mq}$ of $\boldsymbol{u}_m$ such that

$$|\boldsymbol{u}_{mq}-\boldsymbol{u}_0|_1>\varepsilon, \quad \forall\varepsilon>0. \tag{5.17}$$

But $\boldsymbol{u}_{mq}$ is also a minimizing sequence of J which satisfies (5.11). According to the previous discussion we obtain

$$\boldsymbol{u}_{mr}\to\boldsymbol{u}_0 \text{ (strongly)} \quad \text{in } H, \tag{5.18}$$

where $\boldsymbol{u}_{mr}$ is a subsequence of $\boldsymbol{u}_{mq}$. This contradicts assumption (5.17), and we have (5.12).

6. The Construction of a Minimizing Sequence

Let us use the conjugate gradient method to make a minimizing sequence.

1° We take $\boldsymbol{u}_0$ as the solution of the corresponding Stokes equation:

$$A(\boldsymbol{u}_0, \boldsymbol{v})=\langle \boldsymbol{F}, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in H. \tag{6.1}$$

Then compute $\boldsymbol{g}_0\in H$ by

$$A(\boldsymbol{g}_0, \boldsymbol{v})=\langle \mathrm{Grad}\, J(\boldsymbol{u}_0), \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in H, \tag{6.2}$$

and set $\boldsymbol{\zeta}_0=\boldsymbol{g}_0$.

From $m\geqslant 0$, assuming $\boldsymbol{u}_m, \boldsymbol{g}_m, \boldsymbol{\zeta}_m$ are known, compute $\boldsymbol{u}_{m+1}, \boldsymbol{g}_{m+1}, \boldsymbol{\zeta}_{m+1}$ by the following steps:

2° $$\lambda_m=\arg\min_{\lambda\in R} J(\boldsymbol{u}_m-\lambda\boldsymbol{\zeta}_m), \quad \boldsymbol{u}_{m+1}=\boldsymbol{u}_m-\lambda_m\boldsymbol{\zeta}_m. \tag{6.3}$$

3° Find $\boldsymbol{g}_{m+1}\in H$ such that

$$A(\boldsymbol{g}_{m+1}, \boldsymbol{v})=\langle \mathrm{Grad}\, J(\boldsymbol{u}_{m+1}), \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in H, \tag{6.4}$$

$$\gamma_{m+1}=A(\boldsymbol{g}_{m+1}, \boldsymbol{g}_{m+1}-\boldsymbol{g}_m)/A(\boldsymbol{g}_m, \boldsymbol{g}_m), \tag{6.5}$$

$$\boldsymbol{\zeta}_{m+1}=\boldsymbol{g}_{m+1}+\gamma_{m+1}\boldsymbol{\zeta}_m. \tag{6.6}$$

Let $m+1\Rightarrow m$. Go to step 2°.

By means of (3.21) it is easy to check the following lemma:

Lemma 9. *Suppose $J(\boldsymbol{u})$ is defined by (2.7) and (2.8). Then $J(\boldsymbol{u}-t\boldsymbol{v})$ is a polynomial of degree 4 of t: $\forall t\in R$, $\boldsymbol{u}, \boldsymbol{v}\in H$,*

$$2J(\boldsymbol{u}+t\boldsymbol{v})=f(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2+\alpha_3 t^3+\alpha_4 t^4, \tag{6.7}$$

where

$$\begin{cases} \alpha_0=A(\boldsymbol{u}-T\boldsymbol{u}, \boldsymbol{u}-T\boldsymbol{u}), \quad \alpha_1=2A(\boldsymbol{u}-T\boldsymbol{u}, \boldsymbol{v}-T'(\boldsymbol{u})\boldsymbol{v}), \\ \alpha_2=A(\boldsymbol{v}-T'(\boldsymbol{u})\boldsymbol{v}, \boldsymbol{v}-T'(\boldsymbol{u})\boldsymbol{v})-A(\boldsymbol{u}-T\boldsymbol{u}, T'(\boldsymbol{v})\boldsymbol{v}), \\ \alpha_3=A(T'(\boldsymbol{u})\boldsymbol{v}-\boldsymbol{v}, T'(\boldsymbol{v})\boldsymbol{v}), \quad \alpha_4=0.25A(T'(\boldsymbol{v})\boldsymbol{v}, T'(\boldsymbol{v})\boldsymbol{v}). \end{cases} \tag{6.8}$$

It is clear that, $\boldsymbol{\xi}_0=T\boldsymbol{u}$, $\boldsymbol{\xi}_1=T'(\boldsymbol{u})\boldsymbol{v}$, $\boldsymbol{\xi}_2=T'(\boldsymbol{v})\boldsymbol{v}/2$ are the solutions of the following variational problems respectively:

$$\begin{cases} A(\boldsymbol{\xi}_0, \boldsymbol{\eta})=\langle \boldsymbol{F}, \boldsymbol{\eta}\rangle-a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{\eta}), \\ A(\boldsymbol{\xi}_1, \boldsymbol{\eta})=-a_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{\eta})-a_1(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{\eta}), \quad \forall \boldsymbol{\eta}\in H. \\ A(\boldsymbol{\xi}_2, \boldsymbol{\eta})=-a_1(\boldsymbol{v}; \boldsymbol{v}, \boldsymbol{\eta}), \end{cases} \tag{6.9}$$

Theorem 7. *$\forall \boldsymbol{u}, \boldsymbol{v}\in H$, there exists a solution of the single variable minimization problem*

$$t^* = \arg \min_{t \in R} J(\boldsymbol{u} + t\boldsymbol{v})$$

and t^* *is a zero point of polynomial* $f'(t) = df/dt$ *where* $f(t)$ *is defined by* (6.7).

Proof. It is known from Lemma 9 that

$$2J(\boldsymbol{u} + t\boldsymbol{v}) = f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4.$$

1° When $\boldsymbol{\xi}_2 \neq 0$, one has $\alpha_4 = A(\boldsymbol{\xi}_2, \boldsymbol{\xi}_2) > 0$ owing to the coerciveness of $A(\cdot, \cdot)$. In this case $f(t)$ is a polynomial of degree 4. Certainly, $f(t)$ achieves its infimum in the finite interval.

2° When $\boldsymbol{\xi}_2 = 0$, one has $\alpha_4 = 0$. However,

$$\alpha_3 = -A(\boldsymbol{v} - \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = 0,$$

$$\alpha_2 = A(\boldsymbol{v} - \boldsymbol{\xi}_1, \boldsymbol{v} - \boldsymbol{\xi}_1) - 2A(\boldsymbol{u} - \boldsymbol{\xi}_0, \boldsymbol{\xi}_2) = A(\boldsymbol{v} - \boldsymbol{\xi}_1, \boldsymbol{v} - \boldsymbol{\xi}_1).$$

Likewise, when $\boldsymbol{v} - \boldsymbol{\xi}_1 \neq 0$, $\alpha_2 > 0$, $f(t)$ is the polynomial of degree 2, and achieves its infimum in the finite interval.

3° When $\boldsymbol{v} - \boldsymbol{\xi}_1 = 0$, $\boldsymbol{\xi}_2 = 0$, one has $\alpha_4 = \alpha_3 = \alpha_2 = 0$. In this case

$$-\alpha_1 = A(\boldsymbol{u} - \boldsymbol{\xi}_0, \boldsymbol{v} - \boldsymbol{\xi}_1) = 0.$$

So $f(t) = \alpha_0$. Of course, $f(t)$ can achieve its infimum.

$$f'(t) = \alpha_1/2 + \alpha_2 t + (3/2)\alpha_3 t^2 + 2\alpha_4 t^3.$$

According to 1°, 2° and 3° $f'(t)$ is a polynomial of odd degree; hence there exists at least a zero point of $f'(t)$ at which $f(t)$ achieves its infimum.

In practice, Theorem 7 is very important. Due to it, there are many improvements in computational efficiency and accuracy for the conjugate gradient method by reason that to solve the minimum problem (6.3) it only requires to find a root of the equation of degree 3. If we employ an other method, for example, the Fibonacci method the approximate solution would be obtained through successive contraction of the interval; the number of iterations depends upon the precision. For example, the interval is contracted to 1/100, and the number N of iterations is 11. Contracted to 1/1000, $N = 20$. However, during the iteration $J(\boldsymbol{u}_m - \lambda \boldsymbol{\xi}_m)$ must be computed one by one. In order to evaluate J it is necessary to compute a_1 and A, and to solve the Stokes problem once.

Table 1

		$a_1(\cdot; \cdot, \cdot)$	$A(\cdot, \cdot)$	to solve Stokes prob.	interval	error
Fib.	λ_m	12	12	12	$[-1, 1]$	10^{-2}
our		3	0	2	$(-\infty, +\infty)$	10^{-16}
ζ_m		3	1	1		
N^*		6×10^5	6×10^4	6×10^4		

where N^* is the number of multiplicative operators.

7. Numerical Examples

Two codes based on previous methods have been developed. They allow the calculation of the three dimensional viscous flow in the pumps, pipe and lubrical theory.

Numerical experiments have been completed for low and high Reynolds number flow and various values of the penalization parameter.

To examine the effect of the previous method on the accuracy of computed pressure field and velocity field, an analytical example was studied by means of our codes. Its exact solution ($\boldsymbol{u}^*$, p^*) and approximate solution ($\boldsymbol{u}^h$, p^h) are shown in Tables 2 and 3.

Table 2 Comparison between u^* and u^h

	u^h		u^*	error	
	$R_e=5$ $\varepsilon=10^{-5}$	$R_e=1000$ $\varepsilon=10^{-7}$		$R_e=5$	$R_e=1000$
1	0.500	0.567	0.506		
2	0.500	0.554	0.506		
3	0.499	0.540	0.506	0.0463	
4	0.499	0.526	0.506		0.1328
5	0.499	0.620	0.506		
6	0.499	0.616	0.506		
7	0.499	0.612	0.506		
8	0.569	0.637	0.569		
9	0.579	0.632	0.584		
10	0.590	0.528	0.599		
11	0.614	0.624	0.614		
12	0.630	0.620	0.630		
13	0.645	0.616	0.645		
14	0.660	0.612	0.660		

Table 3 Comparison between p^h and p^*

	at node points			at Gaussian points		
	p^h		p^*	p^h		p^*
	$R_e=5$ $\varepsilon=10^{-5}$	$R_e=1000$ $\varepsilon=10^{-7}$		$R_e=5$ $\varepsilon=10^{-5}$	$R_e=1000$ $\varepsilon=10^{-7}$	
1	1.881	1.881	2.000	1.9736	1.9736	1.998
2	1.872	1.872	1.992	1.9690	1.9690	1.994
3	1.865	1.865	1.984	1.9657	1.9655	1.990
4	1.857	1.857	1.976	1.9611	1.9610	1.984
5	1.850	1.850	1.968	1.9578	1.9576	1.983
6	1.842	1.842	1.960	1.9532	1.9531	1.974
7	1.835	1.835	1.952	1.9499	1.9497	1.974
8	2.180	2.180	2.000	1.9453	1.9452	1.970
9	2.171	2.171	1.992	1.9419	1.9418	1.966
10	2.162	2.162	1.984	1.9375	1.9376	1.962
11	2.154	2.154	1.976	1.9341	1.9341	1.958
12	2.145	2.145	1.968	1.9295	1.9294	1.953
13	2.136	2.136	1.960			
14	2.128	2.129	1.952			
error	0.0763			0.0116		

If we calculate the pressure at the nodes by $p_\varepsilon = -\operatorname{div} \boldsymbol{u}^h/\varepsilon$ or $(p_\varepsilon, q) + (\operatorname{div} \boldsymbol{u}^h, q) = 0$, $\forall q \in M_h$ directly, no good result will be obtaind. To our surprise, if we calculate the pressure at Gaussian quadrature points, then we will obtain high accuracy at the nodes by the extrapolation method shown in Table 3. Examining Tables 2 and 3 we see that the approximate accuracy of the pressure is better than that of velocity.

References

[1] M. O. Bristeau, O. Pironneau, R. Glowinski, J. Periaur, P. Perrier, On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite element method (I): Least square formulation and conjugate gradient solution of the continuous problems, *Comp. Meth. Appl. Mech. Eng.*, 17/18 (1979), 618—657.

[2] V. Giraut, P. A. Raviart, Finite Element Approximation for the Navier-Stokes Equation, Lecture Notes in Mathematics, Vol. 749, Springer-Verlag, Berlin, 1980.

[3] R. Teman, Navier-Stokes Equation, North-Holland, Amsterdam, 1977.

[4] Li Kai-tai, Huang Ai-xiang, Ma Yi-chen, Li Du, Optimal control finite element approximation for penalty variational problem of Navier-Stokes equation, *J. Xi'an Jiaotong Univ.*, Vol. 16, No. 1, 1982, 85—88.

[5] Ma Yi-chen, Optimal control finite element algorithm for penalty variational formulation of the Navier-Stokes equation, *J. Math. Reseach and Exposition*, Vol. 3, No. 1, 1983, 130—132.

[6] J. N. Reddy, On the Mathematical Theory of the Penalty Finite Elements for Navier-Stokes Equations, Proceedings of the Third International Conference on Finite Elements in Flow Problems, Vol. II, 1980.

[7] O. C. Zienkiewicz, Constrained Variational Principles and Penalty Function Methods in Finite Element Analysis, Lecture Notes in Mathematics, 363, Springer-Verlag, New York, 1974.

[8] R. S. Falk, J. T. King, A penalty and extrapolation methcd for the stationary Stokes equations, *SIAM J. Numer. Anal.*, 13 (1979), 814—829.

[9] M. Bercoviex, M. Engelman, A finite element for the numerical solution of viscous incompressible flows, *J. Comp. Phys.*, 30 (1979), 181—201.

[10] T. J. R. Hughes, W. K. Liu, A. Brookes, Finite element analysis of incompressible viscous flows by the penalty function formulation, *J. Comp. Phys.*, 30 (1979), 1—60.

[11] Y. J. Song, I. T. Oden, N. Kikuchi, Discrete LBB-Conditions for RIP-Finite Element Methods, TICON Report 80-7 (1980).

[12] J. T. Oden, RIP-Methods for Stokesian Flows, Finite Elements in Fluids, Vol. 4, John Willey & Sons.

[13] Li Kai-tai, Mixed boundary value problem for Navier-Stokes equation, Scientific and Technical Report 82-306, Xi'an Jiaotong University.