NONCONFORMING ELEMENTS IN THE MIXED FINITE ELEMENT METHOD*

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Abstract

In this paper, an abstract error estimate of mixed finite element methods using nonconforming elements is presented. In addition, a class of nonconforming rectangular elements is proposed, and applied to Stokes equations. The optimal error estimate is given.

I. Introduction

The mixed finite element method has been applied to many difference fields, such as solid mechanics, fluid mechanics, and so on. The study of this method can be reduced to the following abstract saddle-point problem. Let V and W be two real Hilbert spaces, whose norms, scalar products and dual spaces are denoted by $\|\cdot\|_V$, $(\cdot, \cdot)_V$, V' and $\|\cdot\|_W$, $(\cdot, \cdot)_W$, W' respectively. Let $\langle\cdot, \cdot\rangle$ denote duality between both V' and V and W' and W. The variational problem is

Find $(u, p)\in V\times W$, such that

$$\begin{cases} a(u, v)+b(v, p)=\langle f, v\rangle, & \forall v\in V, \\ b(u, q)=\langle g, q\rangle, & \forall q\in W, \end{cases} \tag{1.1}$$

where $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are bounded bilinear forms on $V\times V$ and $V\times W$, respectively, and $f\in V'$, $g\in W'$ are given.

Given two finite dimensional subspaces $V_h\subset V$ and $W_h\subset W$, $0<h\leqslant h_0$, the finite element approximation of (1.1) is the solution of the following problem:

Find $(u_h, p_h)\in V_h\times W_h$, such that,

$$\begin{cases} a(u_h, v)+b(v, p_h)=\langle f, v\rangle, & \forall v\in V_h, \\ b(u_h, q)=\langle g, q\rangle, & \forall q\in W_h. \end{cases} \tag{1.2}$$

In 1974, Brezzi[1] studied the saddle-point problem (1.1) and its finite element approximation. The main results are the following:

Let $Z=\{v\in V;\ b(v, q)=0,\ \forall q\in W\}$. If

(i) there is a constant $\alpha>0$, such that

$$a(v, v)\geqslant\alpha\|v\|_V^2, \quad \forall v\in Z, \tag{1.3}$$

(ii) there exists a constant $\beta>0$, such that

$$\operatorname*{Inf}_{q\in W\setminus\{0\}}\ \operatorname*{Sup}_{v\in V\setminus\{0\}} \frac{b(v, q)}{\|v\|_V\|q\|_W}\geqslant\beta, \tag{1.4}$$

then problem (1.1) has a unique solution (u, p).

Let $Z_h=\{v\in V_h;\ b(v, q)=0,\ \forall q\in W_h\}$. If

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(i) Z_h is not empty,

(ii) there is a constant $\alpha^*>0$ independent of h, such that

$$a(v, v)\geqslant\alpha^*\|v\|_V^2, \quad \forall v\in Z_h, \tag{1.3}_h$$

(iii) there exists a constant $\beta^*>0$ independent of h, such that

$$\operatorname*{Inf}_{q\in W_h\setminus\{0\}}\ \operatorname*{Sup}_{v\in V_h\setminus\{0\}} \frac{b(v, q)}{\|v\|_V\|q\|_W}\geqslant\beta^*, \tag{1.4}_h$$

then problem (1.2) has a unique solution, and the following error estimates hold:

$$\|u-u_h\|_V+\|p-p_h\|_W\leqslant C\{\operatorname*{Inf}_{v_h\in V_h}\|u-v_h\|_V+\operatorname*{Inf}_{q_h\in W_h}\|p-q_h\|_W\}, \tag{1.5}$$

where C is a constant independent of h.

To get the optimal error estimates of the mixed finite element approximation of problem (1.1), we must choose finite dimensional subspaces V_h and W_h carefully. Firstly, V_h and W_h should approximate V and W with the same order of precision. Secondly, V_h and W_h must satisfy the compatibility condition $(1.4)_h$. Babuska[2] studied finite element approximation of general variationally posed problems. The major results of [2] apply to problems (1.1) and (1.2); two similar conditions equivalent to (1.4) and $(1.4)_h$ respectively can be obtained. In general, conditions (1.4) and $(1.4)_h$ are called Babuska–Brezzi conditions.

The boundary value problem of Stokes equations is a model of the abstract variational problem (1.1). Suppose $\Omega\in\mathbb{R}^2$ is a rectangular domain with boundary Γ. We consider the boundary value problem of Stokes equations:

$$\begin{cases} -\mu\Delta u+\operatorname{grad}p=f, & \text{on } \Omega,\\ \operatorname{div}u=0, & \text{on } \Omega,\\ u|_\Gamma=0, \end{cases} \tag{1.6}$$

where $u=(u_1, u_2)$ is the velocity vector, p is the pressure, μ is a positive constant, the coefficient of kinematic viscosity. Let $V=H_0^1(\Omega)\times H_0^1(\Omega)$, $W=H^0(\Omega)/P_0(\Omega)$, where $H_0^1(\Omega)$, $H^0(\Omega)$ denote the usual Sobolev spaces on Ω, and $P_0(\Omega)$ denotes the space of all constants on Ω. Then the boundary value problem (1.6) is equivalent to the following variational problem:

Find $(u, p)\in V\times W$, such that

$$\begin{cases} a(u, v)+b(v, p)=\langle f, v\rangle, & \forall v\in V,\\ b(u, q)=0, & \forall q\in W, \end{cases} \tag{1.7}$$

where $$a(u, v)=\mu\ (\operatorname{grad}u, \operatorname{grad}v)_\Omega\equiv\mu\sum_{i,j=1}^{2}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)_{H^0(\Omega)},$$

$$b(v, p)=-(\operatorname{div}v, p)_\Omega\equiv-(\operatorname{div}v, p)_{H^0(\Omega)}.$$

For example, Ω is divided into nine equal rectangular elements. The subspace of velocity field V_h is formed by piecewise bilinear functions and the subspace of pressure W_h is formed by piecewise constants, but up to now, it is not known whether subspaces V_h and W_h satisfy the B–B condition or not[3]. Some finite element schemes satisfying the B–B condition have been proposed by many authors. For instance, quadratic conforming triangular elements for the velocity field and piecewise constant triangular elements for the pressure were used by Crouzeix and Raviart[4]. Their corresponding subspaces V_h and W_h satisfy the B–B condition, but with a loss of precision of the finite element approximation; and the optimal error estimate cannot be obtained. If we use conforming linear triangular elements for the velocity field, and constant triangular elements for the pressure, then the corresponding subspaces V_h and W_h do not satisfy the B-B condition. Hence, Crouzeix and Raviart proposed to use the nonconforming linear triangular elements to form the approximation space of velocity field for solving Stokes equations, and gave an optimal error estimate. A few years later, the nonconforming triangular elements were applied to Navier-Stokes equations[5,6]. Recently Zhou Tianxiao[7] studied the abstract saddle-point problem (1.1), and gave two criteria of strong Bobuska-Brezzi condition. In this paper, we emphasize the use of nonconforming elements in the mixed finite element method. In section II, an abstract error estimate of mixed element approximation using nonconforming elements is given. This result can be understood as an extension of second Strang lemma[8,10] to mixed finite element methods or an extension of the abstract error estimate for mixed finite element methods given by Falk and Osborn[9]. In section III, we design a class of nonconforming rectangular elements, and apply them to solve stokes equations. The optimal error estimate is given in the last section.

II. An Abstract Error Estimate

In this section, we discuss nonconforming finite element approximation to problem (1.1). Let V_h, W_h be two finite dimensional spaces, and $W_h\subset W$; but, in general, V_h is not a subspace of V. Suppose that H is a real Hilbert space, $V\subset H$, and $V_h\subset H$. We extend the definitions of $a(u, v)$ and $b(v, q)$ to $(V_h\cup V)\times(V_h\cup V)$ and $(V_h\cup V)\times W_h$. Let $a_h(u, v)$ and $b_h(v, q)$ denote those extensions, and

$$\begin{cases} a_h(u, v)=a(u, v), & \forall u, v\in V, \\ b_h(v, q)=b(v, q), & \forall v\in V, q\in W. \end{cases} \tag{2.1}$$

For $v_h\in V_h$, let $\|v_h\|_h$ denote the norm of v_h on space V_h; and suppose $\|v_h\|_h=\|v_h\|_V$, $\forall v_h\in V$. The nonconforming finite element approximation (u_h, p_h) to (u, p) is the solution of the following problem:

Find $(u_h, p_h)\in V_h\times W_h$, such that

$$\begin{cases} a_h(u_h, v_h)+b_h(v_h, p_h)=\langle f, v_h\rangle, & \forall v_h\in V_h, \\ b_h(u_h, q_h)=\langle g, q_h\rangle, & \forall q_h\in W_h. \end{cases} \tag{2.2}$$

From now on, suppose that $f\in H'$. We have the following abstract error estimate:

Theorem 2.1. *Let*

$$Z_h=\{v_h\in V_h;\ b_h(v_h, q_h)=0,\ \forall q_h\in W_h\}.$$

If the following conditions hold:

(i) *there is a constant $\alpha>0$ independent of h, such that*

$$\alpha\|v_h\|_h^2\leqslant a_h(v_h, v_h),\quad \forall v_h\in Z_h, \tag{2.3}$$

(ii) *there exist two constants $A>0$, $B>0$, independent of h, such that*

$$|a_h(u_h, v_h)|\leqslant A\|u_h\|_h\|v_h\|_h,\quad \forall u_h, v_h\in V_h, \tag{2.4}$$

$$|b_h(v_h, q_h)|\leqslant B\|v_h\|_h\|q_h\|_W,\quad \forall v_h\in V_h, q_h\in W_h, \tag{2.5}$$

(iii) *there exists a constant $\beta>0$ independent of h, such that*

$$\sup_{v_h\in V_h\setminus\{0\}}\frac{b_h(v_h,\ q_h)}{\|v_h\|_h}\geqslant\beta\|q_h\|_W,\quad \forall q_h\in W_h, \tag{2.6}$$

(iv) *there is an operator* Π_h: $V\to V_h$, *such that*

$$b_h(u-\Pi_h u,\ q_h)=0,\quad \forall q_h\in W_h, \tag{2.7}$$

where $(u,\ p)$ *is the solution of* (1.1), *then problem* (2.2) *has a unique solution* $(u_h,\ p_h)$; *moreover, the following error estimates hold:*

$$\|u-u_h\|_h\leqslant C\left\{\|u-\Pi_h u\|_h+\inf_{q_h\in W_h}\|p-q_h\|_W+\sup_{v_h\in V_h\setminus\{0\}}\frac{|E_h(u,\ p;\ v_h)|}{\|v_h\|_h}\right\}, \tag{2.8}$$

$$\|p-p_h\|_W\leqslant C\left\{\|u-\Pi_h u\|_h+\inf_{q_h\in W_h}\|p-q_h\|_W+\sup_{v_h\in V_h\setminus\{0\}}\frac{|E_h(u,\ p;\ v_h)|}{\|v_h\|_h}\right\}, \tag{2.9}$$

where C *is a positive constant independent of* h, *and*

$$E_h(u,\ p;\ v_h)=a_h(u,\ v_h)+b_h(v_h,\ p)-\langle f,\ v_h\rangle. \tag{2.10}$$

Proof. Equality (2.10) can be rewritten as

$$a_h(u,\ v_h)+b_h(v_h,\ p)=\langle f,\ v_h\rangle+E_h(u,\ p;\ v_h);\quad \forall v_h\in V_h, \tag{2.10$'$}$$

and combining the first equation of discrete problem (2.2), we obtain

$$a_h(u_h,\ v_h)=a_h(u,\ v_h)+b_h(v_h,\ p-p_h)-E_h(u,\ p;\ v_h),\quad \forall v_h\in V_h. \tag{2.11}$$

Let $\omega_h=u_h-\Pi_h u$. Then $\omega_h\in Z_h$ and we have

$$\begin{aligned}\alpha\|\omega\|_h^2\leqslant a_h(\omega_h,\ \omega_h)&=a_h(u_h,\ \omega_h)+a_h(-\Pi_h u,\ \omega_h)\\&=a_h(u-\Pi_h u,\ \omega_h)+b_h(\omega_h,\ p-p_h)-E_h(u,\ p;\ \omega_h).\end{aligned}$$

On the other hand, by the second equation of discrete problem (2.2), we have

$$b_h(u_h,\ q_h)=\langle g,\ q_h\rangle,\quad \forall q_h\in W_h,$$

and $$b_h(\Pi_h u,\ q_h)=b_h(u,\ q_h)=b(u,\ q_h)=\langle g,\ q_h\rangle,\quad \forall q_h\in W_h$$

by condition (2.7) and (2.1), which yield

$$\|\omega_h\|_h^2\leqslant\frac{1}{\alpha}\{a_h(u-\Pi_h u,\ \omega_h)+b_h(\omega_h,\ p-q_h)-E_h(u,\ p;\ \omega_h)\},\quad \forall q_h\in W_h.$$

Furthermore, we obtain

$$\|u_h-\Pi_h u\|_h\leqslant\frac{1}{\alpha}\left\{A\|u-\Pi_h u\|_h+B\inf_{q_h\in W_h}\|p-q_h\|_W+\sup_{v_h\in V_h\setminus\{0\}}\frac{|E_h(u,\ p;\ v_h)|}{\|v_h\|_h}\right\}. \tag{2.12}$$

The triangle inequality yields

$$\|u-u_h\|_h\leqslant\|u-\Pi_h u\|_h+\|u_h-\Pi_h u\|_h. \tag{2.13}$$

By combining (2.12) and (2.13), inequality (2.8) is shown.

For any $v_h\in V_h$ and $q_h\in W_h$, by equation (2.11) we obtain

$$b_h(v_h,\ p_h-q_h)=a_h(u-u_h,\ v_h)+b_h(v_h,\ p-q_h)-E_h(u,\ p;\ v_h),$$
$$\forall v_h\in V_h,\quad q_h\in W_h.$$

Condition (2.6) yields

$$\|p_h-q_h\|_W\leqslant\frac{1}{\beta}\left\{A\|u-u_h\|_h+B\|p-q_h\|_W+\sup_{v_h\in V_h\setminus\{0\}}\frac{|E_h(u,\ p;\ v_h)|}{\|v_h\|_h}\right\},\quad \forall q_h\in W_h. \tag{2.14}$$

Inequality (2.9) is shown by the triangle inequality.

$$\|p-p_h\|_W \leqslant \|p-q_h\|_W + \|q_h-p_h\|_W$$

and inequalities (2.14) and (2.8).

For the conforming elements, the finite dimensional space V_h is a subspace of space V. Then we know that $E_h(u, p; v_h) \equiv 0$, and error estimates (2.8) and (2.9) reduce to the results given by Falk and Osborn[9].

III. Nonconforming Rectangular Elements

Let $\hat{K}$ be a square, and $\hat{K} = ([-1, 1])^2$. Let $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (-1, 0)$, $\hat{a}_4 = (0, -1)$ denote the middle points of the sides of $\hat{K}$ respectively, and $\hat{a}_5 = (0, 0)$ denote the central point of $\hat{K}$ (as shown in Fig. 1).

Let $\varphi(t) = \frac{1}{2}(5t^4 - 3t^2)$. Then we have

$$\begin{cases} \varphi(0) = 0, \\ \varphi(\pm 1) = 1, \\ \int_{-1}^{1} \varphi(t)\,dt = 0. \end{cases} \tag{3.1}$$

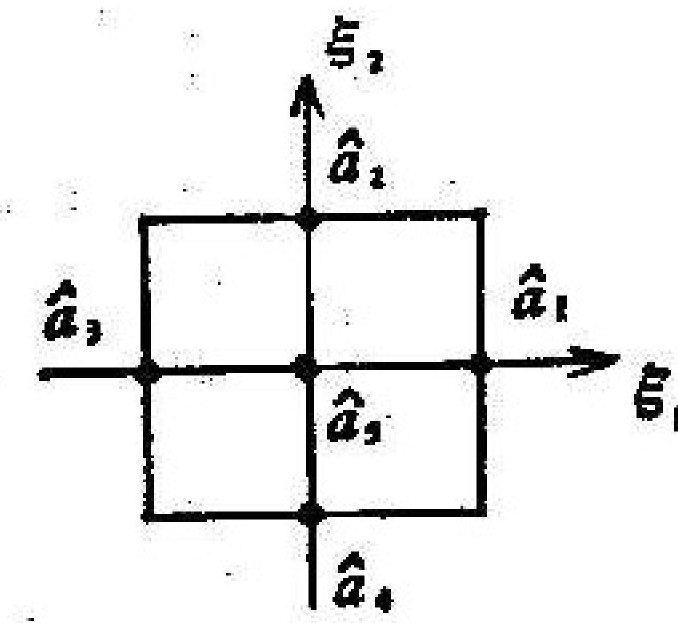

Fig. 1

Furthermore, space $\hat{P}$ is defined by

$$\hat{P} = \{\hat{p} \mid \hat{p} \text{ is a linear combination of } 1,\ \xi_1,\ \xi_2,\ \varphi_1(\xi_1), \varphi_2(\xi_2) \text{ on } \hat{K}\},$$

and we have the following lemma.

Lemma 3.1. *For any five real numbers u_i ($i = 1, 2, \cdots, 5$) there exists a unique function $u(\xi) \in \hat{P}$ ($\xi = (\xi_1, \xi_2)$), such that*

$$u(\hat{a}_i) = u_i, \quad i = 1, 2, \cdots, 5, \tag{3.2}$$

and

$$\begin{cases} \frac{1}{2}\int_{-1}^{1} u(1, \xi_2)\,d\xi_2 = u(1, 0) = u_1, \\ \frac{1}{2}\int_{-1}^{1} u(-1, \xi_2)\,d\xi_2 = u(-1, 0) = u_3, \\ \frac{1}{2}\int_{-1}^{1} u(\xi_1, 1)\,d\xi_1 = u(0, 1) = u_2, \\ \frac{1}{2}\int_{-1}^{1} u(\xi_1, -1)\,d\xi_1 = u(0, -1) = u_4, \\ \frac{1}{4}\int_{\hat{K}} u(\xi)\,d\xi = u(0, 0) = u_5. \end{cases} \tag{3.3}$$

Proof. Let $u(\xi) = A_5 + A_1\xi_1 + A_2\xi_2 + A_3\varphi(\xi_1) + A_4\varphi(\xi_2)$, where constants A_i ($i = 1, 2, \cdots, 5$) are to be determined. By conditions (3.2), we obtain the following linear equations to determine constants A_i

$$\begin{cases} A_5 + A_1 + A_3 = u_1, \\ A_5 + A_2 + A_4 = u_2, \\ A_5 - A_1 + A_3 = u_3, \\ A_5 - A_2 + A_4 = u_4, \\ \qquad\qquad\ A_5 = u_5. \end{cases} \tag{3.4}$$

It is straight forward to get the unique solution of linear equations (3.4)

$$\begin{cases} A_1=\dfrac{u_1-u_3}{2}, \\ A_2=\dfrac{u_2-u_4}{2}, \\ A_3=\dfrac{u_1+u_3-2u_5}{2}, \\ A_4=\dfrac{u_2+u_4-2u_5}{2}, \\ A_5=u_5. \end{cases}$$

Finally,

$$u(\xi)=u_5+\frac{u_1-u_3}{2}\xi_1+\frac{u_2-u_4}{2}\xi_2+\frac{u_1+u_3-2u_5}{2}\varphi(\xi_1)+\frac{u_2+u_4-2u_5}{2}\varphi(\xi_2),$$

and a simple calculation yields equalities (3.3).

Suppose K is an arbitrary rectangle. Let a_i $(i=1, \cdots, 4)$ denote the middle points of the sides of K, a_5 denote the central point of K and l_i $(i=1, \cdots, 4)$ the sides of K. Furthermore, suppose $h_i=\text{meas}\{l_i\}$ $(i=1, \cdots, 4)$ and $a_i=(x_i^1, x_i^2)$ $(i=1, 2, \cdots, 5)$. Then there exists an invertible linear mapping F_K: $\xi\in\hat{K}\to x\in K$, and $K=F_K(\hat{K})$, $a_i=F_K(\hat{a}_i)$, $i=1, 2, \cdots, 5$. Let $P_K=\{p(x)=\hat{p}(F_K^{-1}(x)),\ \hat{p}\in\hat{P}$, and $\Sigma_K=\{p(a_i), 1\leqslant i\leqslant 5\}$. Then Lemma 3.1 implies that for any five real numbers p_i $(i=1, \cdots, 5)$, there is a unique function $p(x)\in P_K$, such that

$$p(a_i)=p_i, \quad i=1, \cdots, 5,$$

and

$$\frac{1}{h_i}\int_{l_i} p\,dl=p_i, \quad i=1, \cdots, 4,$$

$$\frac{1}{\text{meas}(K)}\int_K p\,dx=p_5.$$

For an arbitrary function $u(x)\in H^1(K)$, the interpolating operator Π_K is defined by Π_K: $H^1(K)\to P_K$, such that

$$\frac{1}{h_i}\int_{l_i}\Pi_K u\,dl=\frac{1}{h_i}\int_{l_i}u\,dl, \quad i=1, \cdots, 4,$$

$$\frac{1}{\text{meas}(K)}\int_K \Pi_K u\,dx=\frac{1}{\text{meas}(K)}\int_K u\,dx.$$

For any linear function $u\in P_1(K)$, obviously, we obtain

$$u\equiv\Pi_K u, \quad \forall u\in P_1(K).$$

By the interpolating approximate theorem[10], we have Lemma 3.2. For any $u\in H^2(K)$, and $h_K/\rho_K<\sigma$, there is a constant C independent of h_K and u such that

$$\|u-\Pi_K u\|_{j,K}<Ch_K^{2-j}|u|_{2,K}, \quad j=0, 1, \tag{3.5}$$

where $\sigma>0$ is a constant, $h_K=\max\limits_{1<i<4}\{h_i\}$, $\rho_K=\min\limits_{1<i<4}\{h_i\}$.

Remark. In the three-dimensional case, a nonconforming 3-rectangular element can be constructed in the same manner. We take the six central points of the faces of a 3-rectangular element and the central point of this element as the nodes of this element, and the corresponding space $\hat{P}=\{\hat{p}\,|\,\hat{p}$ is a linear combination of $1, \xi_1, \xi_2, \xi_3, \varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3)\}$. Then the results in Lemma 3.1 and Lemma 3.2 hold for the nonconforming 3-rectangular element.

IV. An Application

Consider the finite element approximation of problem (1.7) using nonconforming rectangular elements. Ω is divided into some subrectangles K, and $\mathscr{T}_K$ denotes this partition satisfying

(i) $\Omega=\bigcup_{K\in\mathscr{T}_K} K$;

(ii) for each distinct K_1 and $K_2\in\mathscr{T}_K$, $K_1\cap K_2$ is empty or a common vertex of K_1 and K_2 or a common side of K_1 and K_2;

(iii) $h=\max_{K\in\mathscr{T}_K}\{h_K\}$; $\rho=\min_{K\in\mathscr{T}_K}\{\rho_K\}$, and

$$h/\rho\leqslant\sigma, \tag{4.1}$$

where σ is a constant.

Let N_0 denote the set of the element nodes, which belong to the boundary of Ω, and N_1 denote the set of all element nodes except those that belong to the boundary of Ω. Furthermore, let $n_1=$ the number of the nodes in set N_1. The finite dimensional space V^h associated with nonconforming rectangular elements (K, P_K, Σ_K) is defined by

$$V^h=\{v\,|\,v|_K\in P_K,\ v(a)=0,\ a\in N_0, \text{ and } v(x) \text{ is continuous at } N_1\}.$$

By Lemma 3.1 we know that V^h is an n_1-dimensional space, and for any $v(x)\in V^h$, $v(x)$ is uniquely determined by the values of $v(x)$ at N_1.

Let $V_h=V^h\times V^h$, and

$$W_h=\left\{q\,|\,q|_K \text{ is a constant, and } \int_\Omega q\,dx=0\right\}.$$

Clearly, W_h is a finite dimensional subspace of W, but V_h is not a subspace of V. Therefore we need to define the approximate bilinear forms $a_h(\cdot,\ \cdot)$ and $b_h(\cdot,\ \cdot)$ on $(V_h\cup V)^2$ and $(V_h\cup V)\times W_h$:

$$a_h(u_h,\ v_h)=\mu\sum_{K\in\mathscr{T}_K}\sum_{i,j=1}^{2}\left(\frac{\partial u_i^h}{\partial x_j}\ \frac{\partial v_i^h}{\partial x_j}\right)_K,\qquad \forall u_h,\ v_h\in V_h, \tag{4.2}$$

$$b_h(v_h,\ p)=-\sum_{K\in\mathscr{T}_K}(\operatorname{div} v_h,\ p)_K,\quad \forall v_h\in V,\ p\in W_h, \tag{4.3}$$

where $(\cdot,\ \cdot)_K$ denotes the scalar product of space $H^0(K)$, $u_h=(u_1^h,\ u_2^h)$, $v_h=(v_1^h,\ v_2^h)$. We obtain the discrete problem:

Find $(u_h,\ p_h)\in V_h\times W_h$, such that

$$\begin{cases} a_h(u_h,\ v_h)+b_h(v_h,\ p_h)=\langle f,\ v_h\rangle, & \forall v_h\in V_h,\\ b_h(u_h,\ q_h)=0, & \forall q_h\in W_h. \end{cases} \tag{4.4}$$

For any $v_h\in V_h$, let $\|v_h\|_h=(a_h(v_h,\ v_h))^{1/2}$; then $\|\cdot\|_h$ is a norm of space V_h. By the definitions of $a_h(\cdot,\ \cdot)$ and $b_h(\cdot,\ \cdot)$, we have

(i) $$\|v_h\|_h^2\leqslant a_h(v_h,\ v_h),\quad \forall v_h\in V_h,$$

(ii) $$|a_h(u_h,\ v_h)|\leqslant A\|u_h\|_h\|v_h\|_h,\quad \forall u_h,\ v_h\in V_h,$$

$$|b_h(v_h,\ q_h)|\leqslant B\|v_h\|_h\|q_h\|_W,\quad \forall v_h\in V_h,\ q_h\in W_h,$$

with $$A=1,\quad B=\sqrt{\frac{2}{\mu}}.$$

Furthermore, operator Π_h: $V\to V_h$ is defined by

$$\Pi_h v(x)=\Pi_K v(x),\quad \forall x\in K,\ K\in\mathscr{T}_K,\quad \forall v\in V.$$

Hence we get

$$b_h(u-\Pi_h u,\ q_h)=\sum_{K\in\mathscr{T}_K}(\operatorname{div}(u-\Pi_K u),\ q_h)_K=\sum_{K\in\mathscr{T}_K}q_h\int_K\operatorname{div}(u-\Pi_K u)dx=0,$$
$$\forall q_h\in W_h.$$

The above discussion shows that conditions (i), (ii) and (iv) of Theorem 2.1 hold. The remaining condition (iii) will be checked in the following lemma.

Lemma 4.1. *There exists a constant $\beta>0$ independent of h, such that*

$$\operatorname*{Sup}_{v_h\in V_h\setminus\{0\}}\frac{b_h(v_h,\ q_h)}{\|v_h\|_h}\geqslant\beta\|q_h\|_W,\quad \forall q_h\in W_h.$$

Proof. For any $q_h\in W_h$, $q_h\in W$ and there is an element $v\in V$[11], such that

$$\operatorname{div} v=-q_h,\quad \|v\|_V\leqslant C\|q_h\|_W. \tag{4.5}$$

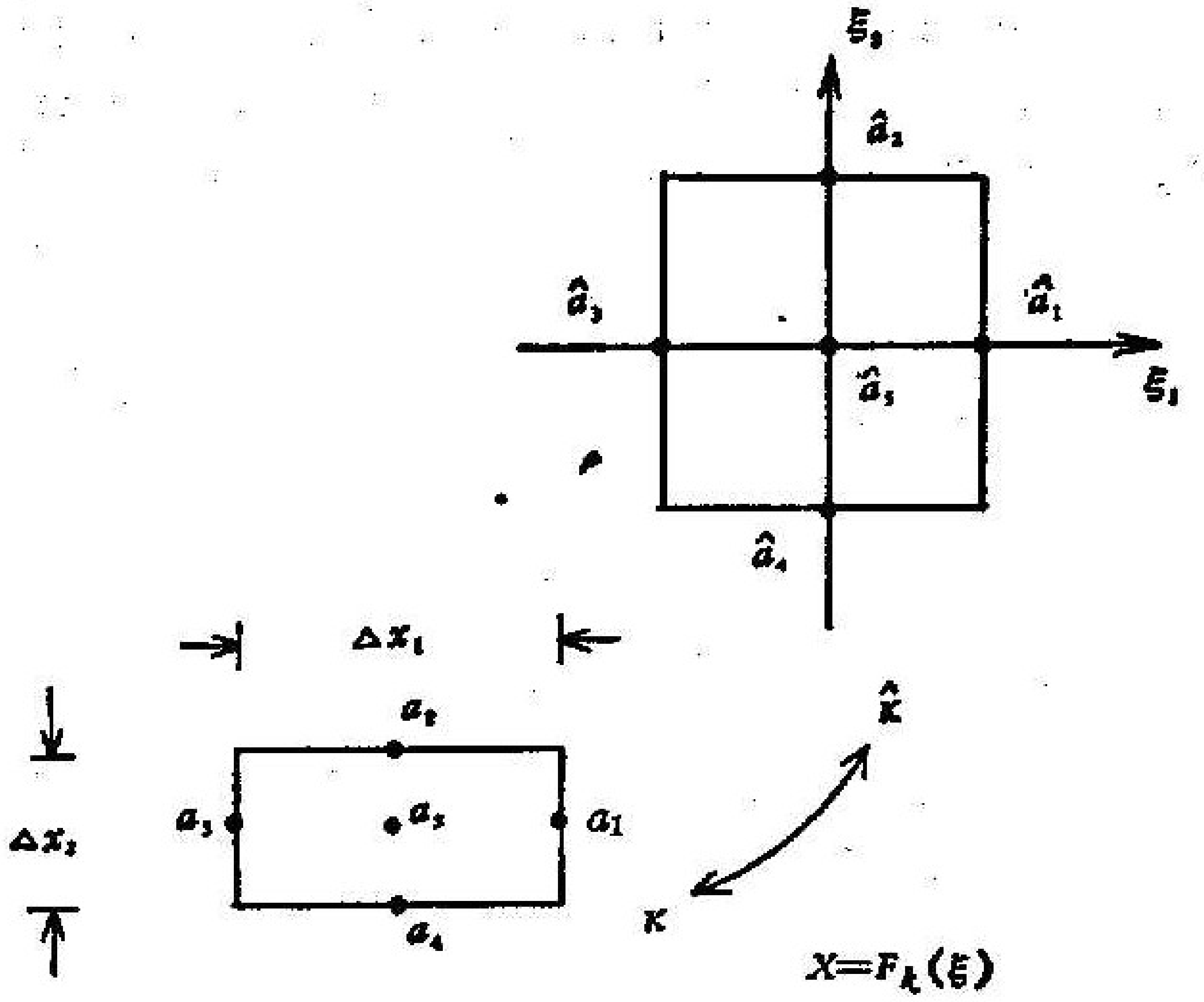

Fig. 2

Here and henceforth, the letter C denotes various constants independent of h.

For any $v\in V$, we have $\Pi_h v\in V_h$. Now we prove the inequality

$$\|\Pi_h v\|_h\leqslant C\|v\|_V,\quad \forall v\in V. \tag{4.6}$$

By the definition of norm $\|\cdot\|_h$, we get

$$\|\Pi_h v\|_h^2=\mu\sum_{K\in\mathscr{T}_K}\sum_{i=1}^{2}\int_K\left(\frac{\partial\Pi_K v_i}{\partial x_1}\right)^2+\left(\frac{\partial\Pi_K v_i}{\partial x_2}\right)^2dx, \tag{4.7}$$

where $v=(v_1,\ v_2)$. We need to estimate each term of the right hand side of equality (4.7). For any $K\in\mathscr{T}_K$, the invertible linear mapping F_K: $\xi\in\hat K\to x\in K$ given by

$$\begin{cases} x_1=\dfrac{1}{2}\,\Delta x_1\xi_1+x_5^1,\\[2mm] x_2=\dfrac{1}{2}\,\Delta x_2\xi_2+x_5^2,\end{cases} \tag{4.8}$$

where $a_5=(x_5^1,\ x_5^2)$, is the central point of K; Δx_1 and Δx_2 denote the length of sides of K respectively, as shown in Fig. 2.

Let $\hat v_i(\xi)=v_i(F_K(\xi))$, $\hat\Pi\hat v_i(\xi)=\Pi_K v_i(F_K(\xi))$. Then we have

$$\int_K\left[\left(\frac{\partial\Pi_h v_i}{\partial x_1}\right)^2+\left(\frac{\partial\Pi_h v_i}{\partial x_2}\right)^2\right]dx\leqslant C\int_{\hat K}\left[\left(\frac{\partial\hat\Pi\hat v_i}{\partial\xi_1}\right)^2+\left(\frac{\partial\hat\Pi\hat v_i}{\partial\xi_2}\right)^2\right]d\xi.$$

By the definition of operator $\hat\Pi$ and Lemma 3.1, we obtain

$$\hat\Pi\hat v_i=A_1\xi_1+A_2\xi_2+A_3\varphi(\xi_1)+A_4\varphi(\xi_2)+A_5,$$

where

$$A_1=\frac{\hat{\Pi}\hat{v}_i(\hat{a}_1)-\hat{\Pi}\hat{v}_i(\hat{a}_3)}{2}=\frac{1}{4}\int_{-1}^{1}[\hat{v}_i(1,\ \xi_2)-\hat{v}_i(-1,\ \xi_2)]d\xi_2=\frac{1}{4}\int_{\hat{K}}\frac{\partial\hat{v}_i}{\partial\xi_1}\,d\xi,$$

$$\begin{aligned}A_3&=\frac{1}{4}\left\{\int_{-1}^{1}[\hat{v}_i(1,\ \xi_2)+\hat{v}_i(-1,\ \xi_2)]d\xi_2-\int_{-1}^{1}\int_{-1}^{1}\hat{v}_i(\xi_1,\ \xi_2)d\xi_1d\xi_2\right\}\\&=\frac{1}{4}\left[\int_{-1}^{1}\hat{v}_i(1,\ \xi_2)d\xi_2-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\hat{v}_i(\xi_1,\ \xi_2)d\xi_1d\xi_2\right]\\&\quad+\frac{1}{4}\left[\int_{-1}^{1}\hat{v}_i(-1,\ \xi_2)d\xi_2-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\hat{v}_i(\xi_1,\ \xi_2)d\xi_1d\xi_2\right],\end{aligned}$$

and
$$A_2=\frac{1}{4}\int_{\hat{K}}\frac{\partial\hat{v}_i}{\partial\xi_2}\,d\xi,$$

$$\begin{aligned}A_4&=\frac{1}{4}\left[\int_{-1}^{1}\hat{v}_i(\xi_1,\ 1)d\xi_1-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\hat{v}_i(\xi_1,\ \xi_2)d\xi_1d\xi_2\right]\\&\quad+\frac{1}{4}\left[\int_{-1}^{1}\hat{v}_i(\xi_1,\ -1)d\xi_1-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\hat{v}_i(\xi_1,\ \xi_2)d\xi_1d\xi_2\right].\end{aligned}$$

Hence,
$$A_1^2\leqslant\int_{\hat{K}}\left(\frac{\partial\hat{v}_i}{\partial\xi_1}\right)^2d\xi,\qquad A_2^2\leqslant\int_{\hat{K}}\left(\frac{\partial\hat{v}_i}{\partial\xi_2}\right)^2d\xi,$$
$$A_3^2\leqslant\int_{\hat{K}}\left(\frac{\partial\hat{v}_i}{\partial\xi}\right)^2d\xi,\qquad A_4^2\leqslant\int_{\hat{K}}\left(\frac{\partial\hat{v}_i}{\partial\xi_2}\right)^2d\xi.$$

By calculation, we obtain

$$\begin{aligned}&\int_{\hat{K}}\left[\left(\frac{\partial\hat{\Pi}\hat{v}_i}{\partial\xi_1}\right)^2+\left(\frac{\partial\hat{\Pi}\hat{v}_i}{\partial\xi_2}\right)^2\right]d\xi\leqslant C(A_1^2+A_2^2+A_3^2+A_4^2)\\&\leqslant C\int_{\hat{K}}\left[\left(\frac{\partial\hat{v}_i}{\partial\xi_1}\right)^2+\left(\frac{\partial\hat{v}_i}{\partial\xi_2}\right)^2\right]d\xi\leqslant C\int_K\left[\left(\frac{\partial v_i}{\partial x_1}\right)^2+\left(\frac{\partial v_i}{\partial x_2}\right)^2\right]dx.\end{aligned}$$

And it yields inequality (4.6).

For any $q_h\in W_h$, first we find a $v\in V$ satisfying (4.5), and combining (4.6) we obtain

$$\begin{aligned}\operatorname*{Sup}_{v_h\in V_h\setminus\{0\}}\frac{b_h(v_h,\ q_h)}{\|v_h\|_h}&\geqslant\frac{b_h(\Pi_hv,\ q_h)}{\|\Pi_hv\|_h}=\frac{b_h(v,\ q_h)}{\|\Pi_hv\|_h}=\frac{b(v,\ q_h)}{\|\Pi v\|_h}\\&=\frac{\|q_h\|_W^2}{\|\Pi_hv\|_h}\geqslant\frac{1}{c}\|q_h\|_W,\quad\forall q_h\in W_h,\end{aligned}$$

which completes the proof.

An application of Theorem 2.1 implies that abstract error estimates (2.8) and (2.9) are valid for discrete problem (4.4). We need to estimate $\operatorname*{Sup}_{v_h\in V_h\setminus\{0\}}\frac{|E_h(u,\ p;\ v_h)|}{\|v_h\|_h}$ to get the concrete error estimates of discrete problem (4.4). We have

Lemma 4.2. *Let $(u,\ p)$ be the solution of problem (1.7), and $u\in V\cup(H^2(\Omega)\times H^2(\Omega))$, $p\in W\cup H^1(\Omega)$. Then there exists a constant C independent of h, such that*

$$|E_h(u,\ p;\ v_h)|\leqslant Ch\{|u|_{2,\Omega}+|p|_{1,\Omega}\}\|v_h\|_h,\quad\forall v_h\in V_h,$$

where $|u|_{2,\Omega}$ and $|p|_{1,\Omega}$ denote the semi-norm of u and p respectively.

Proof. Combining (1.6) and (2.10), we obtain

$$E_h(u,\ p;\ v_h)=\mu\sum_{K\in\mathcal{T}_h}\int_{\partial K}\sum_{i=1}^{2}\left(\frac{\partial u_i}{\partial n}\,v_i^h\right)dl-\sum_{K\in\mathcal{T}_h}\int_{\partial K}(pv_1^hdx_2-pv_2^hdx_1),$$

where $u=(u_1,\ u_2)$, $v_h=(v_1^h,\ v_2^h)$, ∂K denotes the boundary of K, and $\frac{\partial u_i}{\partial n}$ denotes the outward normal derivative of u_i on ∂K.

Let $z(x)=(z_1(x), z_2(x))$, where $z_1(x)$ and $z_2(x)$ are defined only on the boundary of K, $\forall K\in\mathscr{T}_K$. On each side of K, $z_i(x)$ is equal to the value of v_i^h at the middle point of this side. Suppose $\tau(x)=v_h(x)-z(x)$. We have

$$E_h(u, p; v_h)=\mu\sum_{K\in\mathscr{T}_K}\sum_{i=1}^{2}\int_{\partial K}\frac{\partial u_i}{\partial n}\tau_i dl-\sum_{K\in\mathscr{T}_K}\int_{\partial K}(p\tau_1 dx_2-p\tau_2 dx_1). \quad (4.10)$$

We need to estimate each term of (4.10). Let

$$I_1=\int_{\partial K}p\tau_1 dx_2,\quad I_2=\int_{\partial K}p\tau_2 dx_1,\quad I_3^i=\int_{\partial K}\frac{\partial u_i}{\partial n}\tau_i dl.$$

By mapping $x=F_K(\xi)$, and $\hat{v}_i^h(\xi_1, \xi_2)=v_i^h(F_K(\xi))$, $\hat{p}(\xi_1, \xi_2)=p(F_K(\xi))$, integral I_1 can be rewritten as

$$I_1=\frac{\Delta x_2}{2}\int_{-1}^{1}\left[\hat{p}(1, \xi_2)\hat{\tau}_1(1, \xi_2)-\hat{p}(-1, \xi_2)\hat{\tau}_1(-1, \xi_2)\right]d\xi_2.$$

On the other hand, we have

$$\hat{v}_i^h(\xi_1, \xi_2)=A_1^i\xi_1+A_2^i\xi_2+A_3^i\varphi(\xi_1)+A_4^i\varphi(\xi_2)+A_5^i,\quad i=1, 2,$$

$$\hat{\tau}_1(\xi_1, \xi_2)|_{\xi_1=\pm1}=\hat{v}_1^h(\pm1, \xi_2)-\hat{v}_1(\pm1, 0)=A_2^1\xi_2+A_4^1\varphi(\xi_2).$$

That is, $\hat{\tau}_1(1, \xi_2)=\hat{\tau}_1(-1, \xi_2)$. It yields

$$\begin{aligned}|I_1|&=\frac{\Delta x_2}{2}\left|\int_{-1}^{1}[\hat{p}(1, \xi_2)-\hat{p}(-1, \xi_2)]\hat{\tau}(1, \xi_2)d\xi_2\right|\\&=\frac{\Delta x_2}{2}\left|\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\hat{p}(\xi_1, \xi_2)}{\partial\xi_1}\hat{\tau}_1(1, \xi_2)d\xi_1 d\xi_2\right|\\&\leqslant\frac{\Delta x_2}{2}|\hat{p}|_{1,\hat{K}}\sqrt{\int_{\hat{K}}(\hat{\tau}_1(1, \xi_2))^2 d\xi}.\end{aligned}$$

By computation, we get

$$\sqrt{\int_{\hat{K}}(\hat{\tau}_1(1, \xi_2))^2 d\xi}\leqslant C\sqrt{(A_2^1)^2+(A_4^1)^2}\leqslant C|\hat{v}_1^h|_{1,\hat{K}}.$$

Hence, we obtain

$$|I_1|\leqslant Ch_K|p|_{1,\hat{K}}|v_1^h|_{1,\hat{K}}\leqslant Ch_K|p|_{1,K}|v_1^h|_{1,K}.$$

By $h_K\leqslant h$, $\forall K\in\mathscr{T}_K$, we have

$$\left|\sum_{K\in\mathscr{T}_h}\int_{\partial K}(p\tau_1)dx_2\right|\leqslant Ch|p|_{1,\Omega}\|v_h\|_h.$$

Similarly, we can prove

$$\left|\sum_{K\in\mathscr{T}_h}\int_{\partial K}p\tau_2 dx_1\right|\leqslant Ch|p|_{1,\Omega}\|v_h\|_h,$$

$$\left|\mu\sum_{K\in\mathscr{T}_h}\int_{\partial K}\sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial n}\tau_i\right)dl\right|\leqslant Ch|u|_{2,\Omega}\|v_h\|_h.$$

So inequality (4.9) is shown.

Finally, combining the abstract error estimates (2.8), (2.9) and Lemmas 3.2 and 4.2, we have

Theorem 4.1. *Suppose that (u, p) is the solution of problem (1.7) and $u\in V\cup(H^2(\Omega)\times H^2(\Omega))$, $p\in W\cup H^1(\Omega)$, and (u_h, p_h) is the solution of discrete problem (4.4); then there exists a constant C independent of h, u, and p, such that*

$$\|u-u_h\|_h\leqslant Ch\{|p|_{1,\Omega}+|u|_{2,\Omega}\},$$

$$\|p-p_h\|_{0,\Omega}\leqslant Ch\{|p|_{1,\Omega}+|u|_{2,\Omega}\}.$$

References

[1] F. Brezzi, On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers, *RAIRO Numer. Anal.*, 8-R2 (1974), 129—151.

[2] I. Babuška, A. K. Aziz, Survey lectures on the mathematical foundations of the finite element method, in The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations, A. K. Aziz, ed., Academic Press, New York, 1973, 5—359.

[3] M. Fortin, An analysis of the convergence of mixed finite element methods, *RAIRO Numer. Anal.*, 11-R4 (1977), 341—354.

[4] M. Crouzeix, P. A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, *RAIRO Numer. Anal.*, 7-R3 (1973), 33—76.

[5] R. Temam, Theory and Numerical Analysis of the Navier-Stokes Equations, North-Holland, Amsterdam, 1978.

[6] R. Rannacher, On the finite element approximation of nonstationary Navier-Stokes problem, in The Approximation Method for Navier-Stokes Problems, R. Rautmann, ed., Springer Verlag, 1979.

[7] Tianxiao Zhou, Equivalency theorem for "saddle-point" finite element schemes and two criteria of strong Babuska-Brezzi condition, *Scientia Sinica*, 24 (1981), 1190—1206.

[8] G. Stang, Variational crimes in the finite element method, in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, ed., Academic Press, New York, 1973, 689—710.

[9] R. S. Falk, J. E. Osborn, Error estimates for mixed methods, *RAIRO Numer. Anal.*, 14-R3 (1980), 249—277.

[10] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[11] V. Girault, P. A. Raviart, Finite Element Approximation of the Navier-Stokes Equations, Springer Verlag, 1981.