THE SECOND-ORDER FLUID IN CELL (FLIC) METHOD FOR THE ONE-DIMENSIONAL UNSTEADY COMPRESSIBLE FLOW PROBLEMS*

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Abstract

In this paper we suggest a second-order fluid in cell (FLIC) method for the one-dimensional unsteady compressible flow problems. The Numerical result obtained by the present method is compared with the one obtained with the original FLIC method and the exact solution for a shock tube problem.

Introduction

The fluid in cell (FLIC) method[1,2] is one of the most useful difference methods in the computational fluid dynamics. However, as it has only first-order accuracy, it cannot give a satisfactory numerical result in some cases. This paper suggests a second-order fluid in cell (FLIC) method for the one-dimensional unsteady compressible flow problems. The result obtained by the present method is compared with the one obtained with the original FLIC method and the exact solution for the shock tube problem. The comparison demonstrates that the second-order FLIC method is satisfactory.

Second-order Fluid in Cell Method

The equations of one-dimensional unsteady compressible fluid flow may be written in the following forms:

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \tag{2}$$

$$\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+\frac{p}{\rho}\frac{\partial u}{\partial x}=0, \tag{3}$$

where ρ is the density, u is the velocity, p is the pressure and e is the internal energy per unit mass. Assume the gas is polytropic; in that case equation of state is

$$p=(r-1)\rho e, \tag{4}$$

where r is a constant greater than one.

Let Δx and Δt, respectively, denote the spatial increment and the time increment.

* Received November 30, 1983.

The second-order FLIC method calculates the quantities at time $(n+1)\Delta t$ in terms of those at time $n\Delta t$ where n is a number of time step. Within one time step, the new quantities are computed in two phases: First, intermediate values are computed for the velocities and specific internal energy, taking into account the effects of acceleration caused by pressure gradients. Second, transport effects are computed.

Phase 1. By the following two-step method, intermediate values $\tilde{u}_i$ and $\tilde{e}_i$ are obtained:

Step 1:

$$\tilde{\tilde{u}}_i=u_i^n-\frac{\Delta t}{2\Delta x}\frac{1}{\rho_i^n}(p_{i+1/2}^n-p_{i-1/2}^n), \tag{5}$$

$$\tilde{\tilde{e}}_i=e_i^n-\frac{\Delta t}{2\Delta x}p_i^n\frac{1}{\rho_i^n}(u_{i+1/2}^n-u_{i-1/2}^n), \tag{6}$$

$$\tilde{\tilde{p}}_i=(r-1)\rho_i^n\tilde{\tilde{e}}_i. \tag{7}$$

Step 2:

$$\tilde{u}_i=u_i^n-\frac{\Delta t}{\Delta x}\frac{1}{\rho_i^n}(\tilde{\tilde{p}}_{i+1/2}-\tilde{\tilde{p}}_{i-1/2}), \tag{8}$$

$$\tilde{e}_i=e_i^n-\frac{\Delta t}{\Delta x}\frac{\tilde{\tilde{p}}_i}{\rho_i^n}(\tilde{\tilde{u}}_{i+1/2}-\tilde{\tilde{u}}_{i-1/2}). \tag{9}$$

Here $$p_{i\pm1/2}^n=\frac{1}{2}(p_i^n+p_{i\pm1}^n),\quad u_{i\pm1/2}^n=\frac{1}{2}(u_i^n+u_{i\pm1}^n),$$

$$\tilde{\tilde{p}}_{i\pm1/2}=\frac{1}{2}(\tilde{\tilde{p}}_i\pm\tilde{\tilde{p}}_{i\pm1/2}),\quad \tilde{\tilde{u}}_{i\pm1/2}=\frac{1}{2}(\tilde{\tilde{u}}_i\pm\tilde{\tilde{u}}_{i\pm1/2}).$$

Phase 2. Transport effects are now computed. We regard the distributions of intermediate values of ρ_i, $\tilde{u}_i$, $\tilde{e}_i$ in each mesh $(x_{i-1/2}, x_{i+1/2})$ as linear functions, i.e.

$$\begin{cases}\rho=\rho_i+(x-x_i)\dfrac{\Delta_i\rho}{\Delta x},\\ \tilde{u}=\tilde{u}_i+(x-x_i)\dfrac{\Delta_i\tilde{u}}{\Delta x}, \quad x_{i-1/2}<x<x_{i+1/2},\\ \tilde{e}=\tilde{e}_i+(x-x_i)\dfrac{\Delta_i\tilde{e}}{\Delta x},\end{cases} \tag{10}$$

where

$$\begin{cases}x_{i-1/2}=x_i-\dfrac{\Delta x}{2},\\ x_{i+1/2}=x_i+\dfrac{\Delta x}{2},\\ \Delta_i\rho=\dfrac{1}{2}(\rho_{i+1}-\rho_{i-1}),\\ \Delta_i\tilde{u}=\dfrac{1}{2}(\tilde{u}_{i+1}-\tilde{u}_{i-1}),\\ \Delta_i\tilde{e}=\dfrac{1}{2}(\tilde{e}_{i+1}-\tilde{e}_{i-1}).\end{cases} \tag{11}$$

In order to preserve the monotonicity of the numerical solution, the following van Leer monotonicity algorithm[3] is used:

$$(\Delta_i w)_{\text{mono}}=\begin{cases}\min\{\lambda|w_i-w_{i-1}|,\ |\Delta_i w|,\ \lambda|w_{i+1}-w_i|\}\operatorname{sign}(\Delta_i w)\\ \qquad \text{if } \operatorname{sign}(w_i-w_{i-1})=\operatorname{sign}(w_{i+1}-w_i)=\operatorname{sign}(\Delta_i w),\\ 0,\quad \text{otherwise.}\end{cases}\tag{12}$$

Here w denotes ρ, $\tilde{u}$ or $\tilde{e}$. Van Leer used $\lambda=2$, but we use $1\leqslant\lambda\leqslant 2$. In order to simplify the notations, the same notations are used for the results of $\Delta_i\rho$, $\Delta_i\tilde{u}$ and $\Delta_i\tilde{e}$ obtained with the monotonicity algorithm (12).

If $\tilde{u}_{i+1/2}=\frac{1}{2}\left\{\left(\tilde{u}_i+\frac{1}{2}\Delta_i\tilde{u}\right)+\left(\tilde{u}_{i+1}-\frac{1}{2}\Delta_{i+1}\tilde{u}\right)\right\}>0$ and $\tilde{u}^l_{i+1/2}=\tilde{u}_i+\frac{1}{2}\Delta_i\tilde{u}>0$, the fluids flow from cell (i) to cell $(i+1)$. If $\tilde{u}_{i+1/2}<0$ and

$$\tilde{u}^r_{i+1/2}=\tilde{u}_{i+1}-\frac{1}{2}\Delta_{i+1}\tilde{u}<0,$$

the fluids flow from cell $(i+1)$ to cell (i). For other situations, we suppose that the fluids do not flow through the boundary $x_{i+1/2}$.

For $\tilde{u}_{i+1/2}>0$ and $\tilde{u}^l_{i+1/2}>0$, let $l_{i+1/2}$ be the maximum distances between the right boundary $x_{i+1/2}$ of cell i and the fluids position from where the fluids reach the boundary in Δt, as shown in Fig. 1. By use of (10), we have

$$l_{i+1/2}=\frac{\tilde{u}^l_{i+1/2}\Delta t}{1+\Delta t\dfrac{\Delta_i\tilde{u}}{\Delta x}}.\tag{13}$$

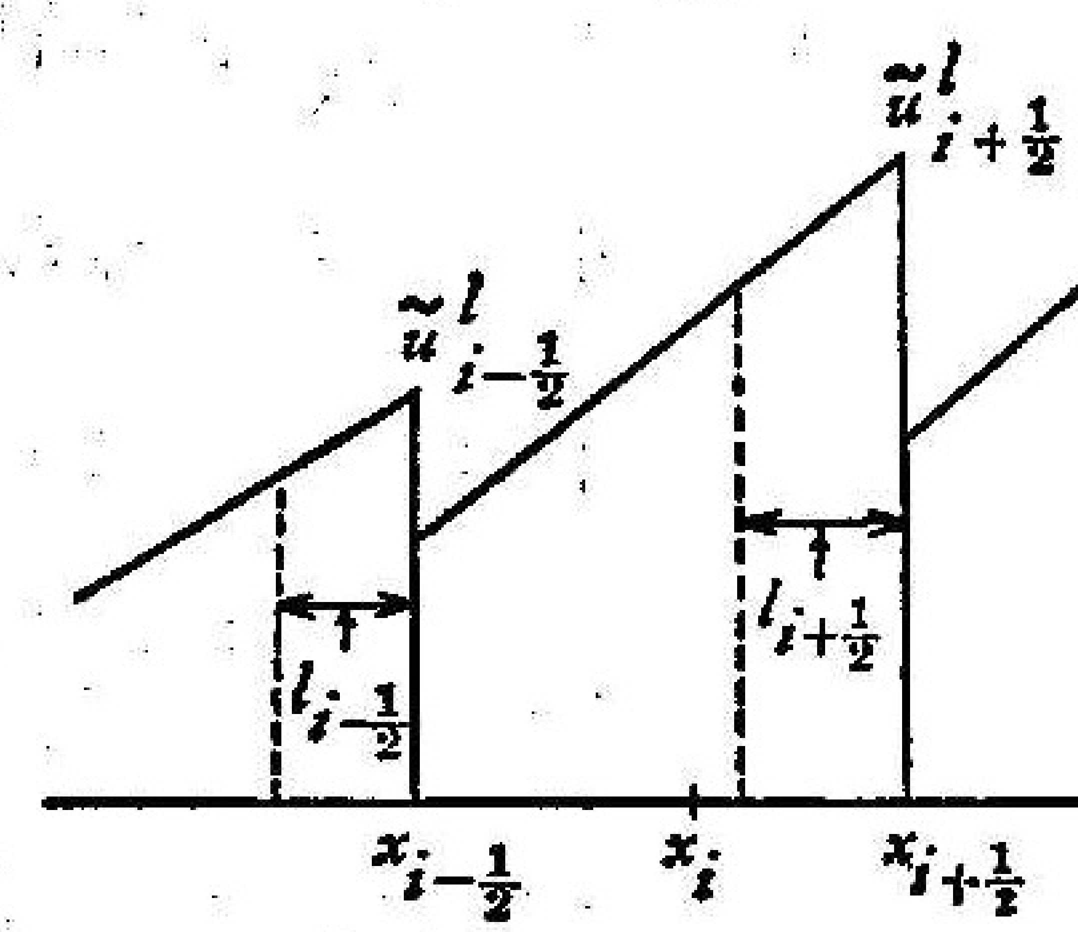


Fig. 1 $\tilde{u}>0$ case

Let $\delta M_{i+1/2}$ be the mass transported from cell (i) to $(i+1)$. Then we have

$$\delta M_{i+1/2}=\int_{x_{i+1/2}-l_{i+1/2}}^{x_{i+1/2}}\left[\rho_i^n+(x-x_i)\frac{\Delta_i\rho}{\Delta x}\right]dx=l_{i+1/2}\left[\rho_i^n+\frac{1}{2}(\Delta x-l_{i+1/2})\frac{\Delta_i\rho}{\Delta x}\right].\tag{14}$$

For the transported momentum $\delta U_{i+1/2}$ we have

$$\begin{aligned}\delta U_{i+1/2}&=\int_{x_{i+1/2}-l_{i+1/2}}^{x_{i+1/2}}\left[\rho_i^n+(x-x_i)\frac{\Delta_i\rho}{\Delta x}\right]\left[\tilde{u}_i+(x-x_i)\frac{\Delta_i\tilde{u}}{\Delta x}\right]dx\\ &=l_{i+1/2}\left\{\rho_i\tilde{u}_i+\left[\rho_i\frac{\Delta_i\tilde{u}}{\Delta x}+\tilde{u}_i\frac{\Delta_i\rho}{\Delta x}\right]\frac{(\Delta x-l_{i+1/2})}{2}\right.\\ &\quad\left.+\frac{\Delta_i\rho}{\Delta x}\frac{\Delta_i\tilde{u}}{\Delta x}\left[\frac{\Delta x^2}{4}+\frac{(\Delta x-l_{i+1/2})}{2}(\Delta x-2l_{i+1/2})\right]\Big/3\right\}.\end{aligned}\tag{15}$$

For the transported energy $\delta E_{i+1/2}$ we have

$$\begin{aligned}\delta E_{i+1/2}&=\int_{x_{i+1/2}-l_{i+1/2}}^{x_{i+1/2}}\left[\rho_i+(x-x_i)\frac{\Delta_i\rho}{\Delta x}\right]\left\{\tilde{e}_i+(x-x_i)\frac{\Delta_i\tilde{e}}{\Delta x}+\frac{1}{2}\left[\tilde{u}_i+(x-x_i)\frac{\Delta_i\tilde{u}}{\Delta x}\right]^2\right\}dx\\ &=l_{i+1/2}\left\{\left[\rho\left(\tilde{e}+\frac{\tilde{u}^2}{2}\right)\right]_i+\left[\frac{\Delta_i\rho}{\Delta x}\left(\tilde{e}+\frac{\tilde{u}^2}{2}\right)_i+\rho_i\left(\frac{\Delta_i\tilde{e}}{\Delta x}+\tilde{u}_i\frac{\Delta_i\tilde{u}}{\Delta x}\right)\right]\frac{(\Delta x-l_{i+1/2})}{2}\right.\\ &\quad+\left[\frac{\Delta_i\rho}{\Delta x}\left(\frac{\Delta_i\tilde{e}}{\Delta x}+\tilde{u}_i\frac{\Delta_i\tilde{u}}{\Delta x}\right)+\frac{\rho_i}{2}\left(\frac{\Delta_i\tilde{u}}{\Delta x}\right)^2\right]\left[\frac{\Delta x^2}{4}+\frac{\Delta x-l_{i+1/2}}{2}(\Delta x-2l_{i+1/2})\right]\Big/3\\ &\quad\left.+\frac{\Delta_i\rho}{\Delta x}\left(\frac{\Delta_i\tilde{u}}{\Delta x}\right)^2\frac{(\Delta x-l_{i+1/2})}{2}\left[\frac{\Delta x^2}{4}+\left(\frac{\Delta x}{2}-l_{i+1/2}\right)^2\right]\Big/4\right\}.\end{aligned}\tag{16}$$

When $\tilde{u}_{i+1/2}<0$ and $\tilde{u}^r_{i+1/2}<0$, we have

$$l_{i+1/2}=\frac{-\tilde{u}^r_{i+1/2}\Delta t}{1+\Delta t\,\dfrac{\Delta_{i+1}\tilde{u}}{\Delta x}} \tag{17}$$

as shown in Fig. 2. For this case, the fluids flow from cell $(i+1)$ to cell (i); then we have

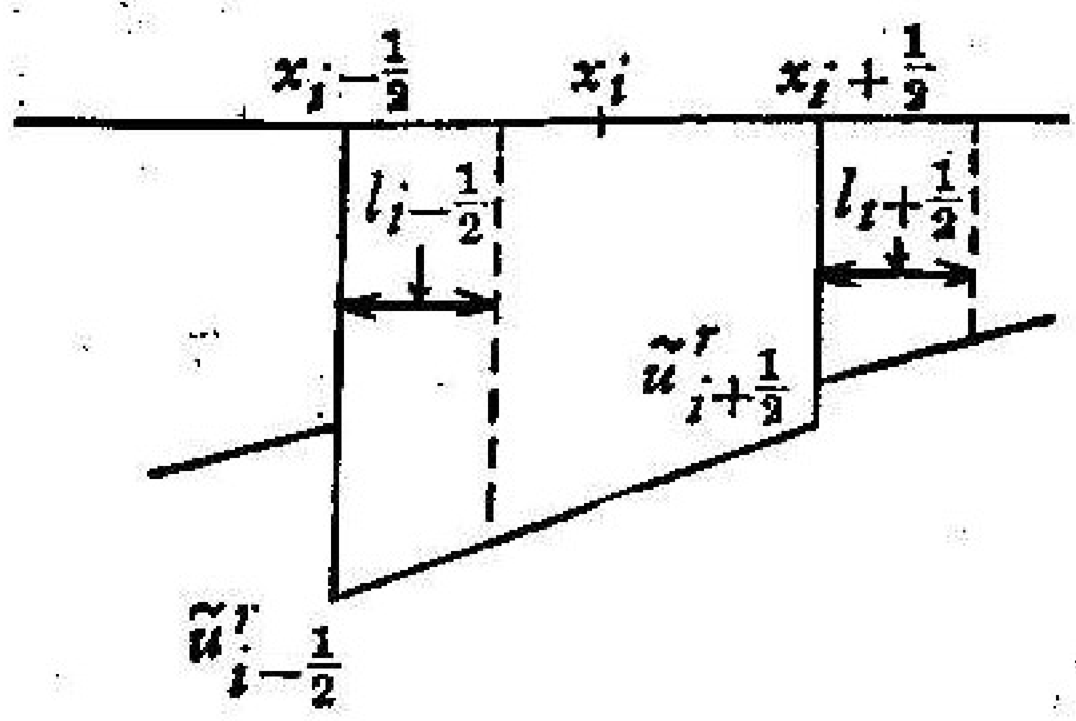


Fig. 2 $\tilde{u}<0$ case

$$\delta M_{i+1/2}=l_{i+1/2}\Big[\rho_{i+1}+\frac{1}{2}(l_{i+1/2}-\Delta x)\frac{\Delta_{i+1}\rho}{\Delta x}\Big], \tag{18}$$

$$\begin{aligned}\delta U_{i+1/2}=l_{i+1/2}\Big\{&\rho_{i+1}\tilde{u}_{i+1}+\Big(\rho_i\frac{\Delta_i\tilde{u}}{\Delta x}+\tilde{u}_i\frac{\Delta_i\rho}{\Delta x}\Big)\\ &\times\frac{(l_{i+1/2}-\Delta x)}{2}+\frac{\Delta_i\rho}{\Delta x}\,\frac{\Delta_i\tilde{u}}{\Delta x}\\ &\times\Big[\frac{(l_{i+1/2}-\Delta x)}{2}(\Delta x-2l_{i+1/2})-\frac{\Delta x^2}{4}\Big]\Big/3\Big\},\end{aligned} \tag{19}$$

$$\begin{aligned}\delta E_{i+1}=l_{i+1/2}\Big\{&\Big[\rho\Big(\tilde{e}+\frac{\tilde{u}^2}{2}\Big)\Big]_{i+1}+\Big[\frac{\Delta_{i+1}\rho}{\Delta x}\Big(\tilde{e}+\frac{\tilde{u}^2}{2}\Big)_{i+1}\\ &+\rho_{i+1}\Big(\frac{\Delta_{i+1}\tilde{e}}{\Delta x}+\tilde{u}_{i+1}\frac{\Delta_{i+1}\tilde{u}}{\Delta x}\Big)\Big]\frac{(l_{i+1/2}-\Delta x)}{2}\\ &+\Big[\frac{\Delta_{i+1}\rho}{\Delta x}\Big(\frac{\Delta_{i+1}\tilde{e}}{\Delta x}+\tilde{u}_{i+1}\frac{\Delta_{i+1}\tilde{u}}{\Delta x}\Big)+\frac{\rho_{i+1}}{2}\Big(\frac{\Delta_{i+1}\tilde{u}}{\Delta x}\Big)^2\Big]\\ &\times\Big[\frac{\Delta x^2}{4}+\frac{(\Delta x-l_{i+1/2})}{2}(\Delta x-2l_{i+1/2})\Big]\Big/3\\ &+\frac{\Delta_{i+1}\rho}{\Delta x}\Big(\frac{\Delta_{i+1}\tilde{u}}{\Delta x}\Big)^2\frac{(l_{i+1/2}-\Delta x)}{2}\Big[\frac{\Delta x^2}{4}+\Big(\frac{\Delta x}{2}-l_{i+1/2}\Big)^2\Big]\Big/4\Big\}.\end{aligned} \tag{20}$$

For the other cases, we have

$$\delta M_{i+1/2}=0,$$
$$\delta U_{i+1/2}=0,$$
$$\delta E_{i+1/2}=0.$$

Similarly, for the left boundary $x_{i-1/2}$ we have $\delta M_{i-1/2}$, $\delta U_{i-1/2}$ and $\delta E_{i-1/2}$.

A new value for the density, the momentum, the total energy in cell (i) respectively now can be obtained by applying the conservation law of mass, momentum and total energy:

$$\rho_i^{n+1}=\rho_i^n+[\mathrm{sign}(\tilde{u}_{i-1/2})\delta M_{i-1/2}-\mathrm{sign}(\tilde{u}_{i+1/2})\delta M_{i+1/2}]/\Delta x, \tag{21}$$

$$\rho_i^{n+1}u_i^{n+1}=\rho_i^n\tilde{u}_i^n+[\mathrm{sign}(\tilde{u}_{i-1/2})\delta U_{i-1/2}-\mathrm{sign}(\tilde{u}_{i+1/2})\delta U_{i+1/2}]/\Delta x, \tag{22}$$

$$\begin{aligned}\rho_i^{n+1}\Big(e_i^{n+1}+\frac{1}{2}(u_i^{n+1})^2\Big)=\rho_i^n\Big(\tilde{e}_i+\frac{1}{2}(\tilde{u}_i)^2\Big)+[&\mathrm{sign}(\tilde{u}_{i-1/2})\delta E_{i-1/2}\\ &-\mathrm{sign}(\tilde{u}_{i+1/2})\delta E_{i+1/2}]/\Delta x.\end{aligned} \tag{23}$$

Hence, we can obtain ρ_i^{n+1}, u_i^{n+1} and e_i^{n+1}.

In order to prevent the numerical oscillations Smagin and Fursenko's monotonicity algorithm[4] is used after phase 2:

$$(W_i)_{\mathrm{mono}}=W_i^{n+1}+\frac{1}{8}(\Delta_{i+1/2}W^{n+1}-\Delta_{i-1/2}W^{n+1}), \tag{24}$$

where

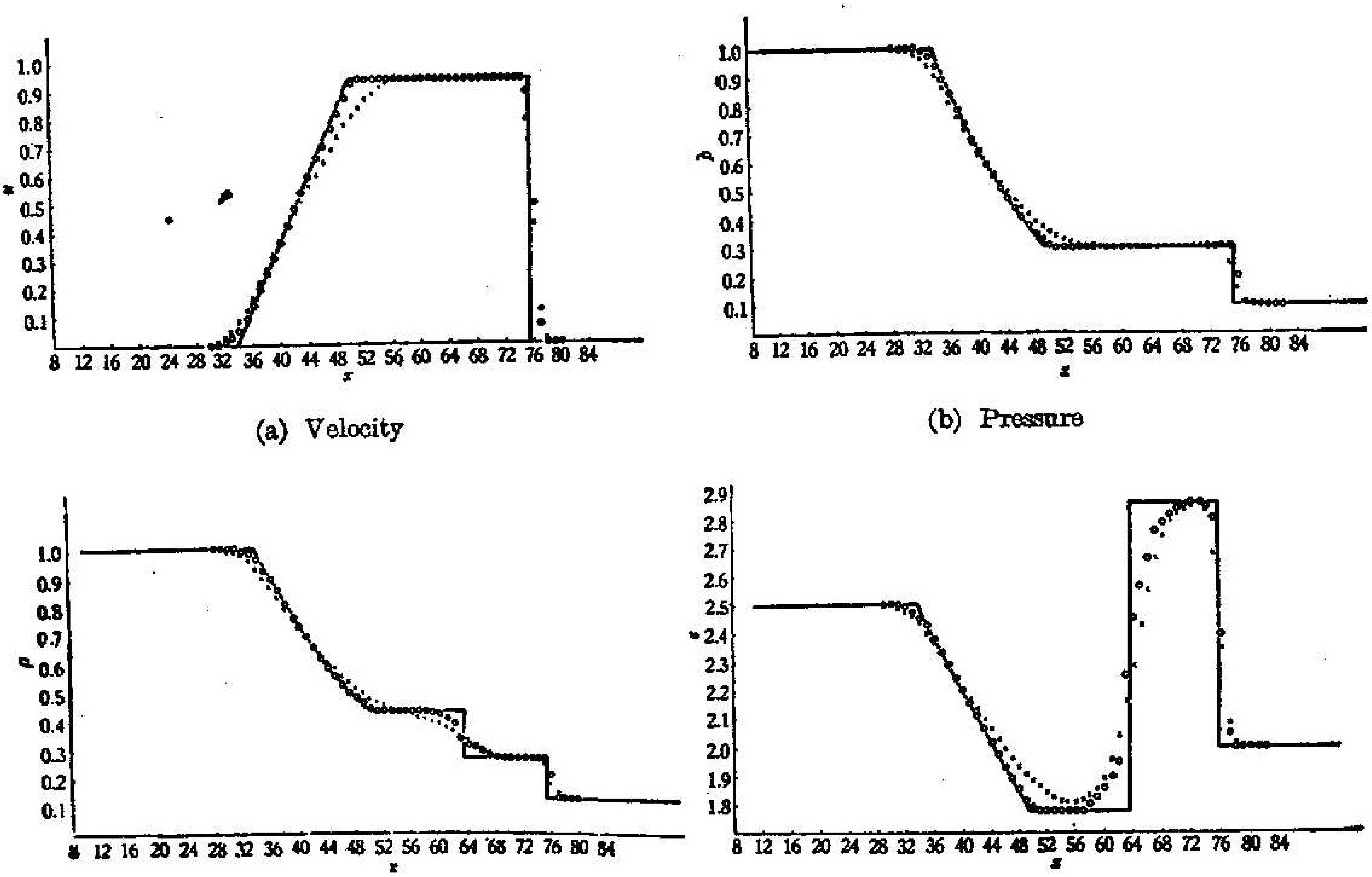
$$\Delta_{i+1/2}W = \begin{cases} W_{i+1}-W_i & \text{if } ((W_{i+1}-W_i)(W_{i+2}-W_{i+1})<0) \\ & \text{or } ((W_{i+1}-W_i)(W_i-W_{i-1})<0), \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

W denotes ρ, u or $\tilde{e}$. A new value for the pressure now can be obtained by applying the equation of state (4).

From a truncation error analysis of the second-order FLIC method described previously, we can show that it has second-order accuracy.

Numerical Example

To illustrate the performance of the method described previously, we compute the same shock tube problem as Sod[5] used for testing various schemes. The tube extends from $x=0$ to $x=1$, and is divided in 100 computational cells. The gas is initially at rest, while at $x=0.5$ the density and the pressure jump from 1 down to 0.125 and 0.1, respectively. The ratio of specific heats is chosen to bo 1.4.



(a) Velocity (b) Pressure

(c) Density (d) Energy

Fig. 3 Numerical solutions of u, p, ρ, e obtained with the present method (by '∘') and those with the original FLIC (by '×') are compared with exact solutions (lines). $\Delta x=0.01$, $t=40\ \Delta t=0.14154$.

The numerical solution of the present method (by "∘") and the original FLIC method (by "×") and the exact solution (by solid line) are shown in Fig. 3 at the time ($t=0.14154=40\Delta t$) when the shock wave moving to the right has approximately reached $x=0.75$. The figure indicates that the second-order FLIC scheme is much better than the original one. Particularly, the rarefaction wave is quite accurate and the result with the present method is even better than two-step Lax-Wendroff's or MacCormack's[5]. The transition of the shock occupies only two cells. The constant state between the contact discontinuity and the shock wave is obviously realized, though the transition of the contact discontinuity occupies still seven to eight cells.

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