THE MONOTONICITY PROBLEM IN FINDING ROOTS OF POLYNOMIALS BY KUHN'S ALGORITHM*

XU SEN-LIN (徐森林)
(China University of Science and Technology)

WANG ZE-KE (王则珂)
(Zhongshan University)

Abstract

In this paper the problem proposed by Kuhn on the presence of a monotonicity property related to the Kuhn's algorithm for finding roots of a polynomial is solved in the affirmative. Furthermore, an estimate of the threshold number D in the above-mentioned monotonicity problem expressed in terms of the complex coefficients of the polynomial is obtained.

Introduction

Kuhn has constructed in [1] the sequences (z_{jk}, d_{jk}), $j=1, \cdots, n$, $k=1, 2, \cdots$; $\lim\limits_{k\to+\infty} z_{jk}=\tilde{z}_j$; $\tilde{z}_1, \cdots, \tilde{z}_n$ are the roots of a monic polynomial $f(z)$ of degree n in the complex variable z with complex numbers as coefficients. He recently posed a monotonicity problem: If $\tilde{z}_1, \cdots, \tilde{z}_n$ are the simple roots of $f(z)$, does there exist a number D such that when $d_{jk}\geqslant D$, both d and $d_{jk}+d_{jk'}+d_{jk''}+d_{jk'''}$ are increasing; (z_{jk}, d_{jk}) belongs to a tetrahedron $\{(z_{jk}, d_{jk}), (z_{jk'}, d_{jk'}), (z_{jk''}, d_{jk''}), (z_{jk'''}, d_{jk'''})\}$, $k>k', k'', k'''$ and $d\leqslant d_{jk}, d_{jk'}, d_{jk''}, d_{jk'''}\leqslant d+1$.

He further asked how to find the expression of D in a_j, $a_1, \cdots, a_n$ being the complex coefficients of $f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$, and how to find D such that when $d\geqslant D$, there is just one triangle labelled (1, 2, 3) in $U((\tilde{z}_j, L)\subset C_d$, where $U(\tilde{z}_j, L)$, $j=1, \cdots, n$, are disjoint open circular discs.

This paper aims to answer these problems.

1. A Monotonicity Problem

Lemma 1.1. *If* $|z|>\max\limits_k|a_k|+1$, *then* $f(z)\neq 0$, *that is*, $\max\limits_k|\tilde{z}_k|\leqslant\max\limits_k|a_k|+1$, *where* $\tilde{z}_1, \cdots, \tilde{z}_n$ *are the roots of* $f(z)$.

Proof. Since

$$|f(z)|=\left|z^n\left(1+\sum_{l=1}^n\frac{a_l}{z^l}\right)\right|\geqslant|z^n|\left(1-\sum_{l=1}^n\frac{|a_l|}{|z|^l}\right)$$

$$\geqslant|z^n|\left(1-\max_k|a_k|\sum_{l=1}^\infty\frac{1}{|z|^l}\right)=|z^n|\left(1-\frac{\max\limits_k|a_k|}{|z|-1}\right)>0,$$

* Received August 28, 1982. The Chinese version was received April 13, 1982.

therefore
$$f(z)\neq 0.$$

Let
$$\varphi(z)=\sum_{l=0}^{n}|a_l|z^l,$$
$$R=\max_k|a_k|+1,$$
$$M=1+\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{(l-1)!}.$$

Then
$$|f^{(s)}(\tilde{z}_j)|=\left|\sum_{l=s}^{n}l(l-1)\cdots(l-s+1)a_l\tilde{z}_j^{l-s}\right|$$
$$\leqslant\sum_{l=s}^{n}l(l-1)\cdots(l-s+1)|a_l|R^{l-s}=\varphi^{(s)}(R),$$
$s=1,\ \cdots,\ n.$

Lemma 1.2. *Let $\tilde{z}_1,\ \cdots,\ \tilde{z}_n$ be the simple roots of $f(z)$; $0<N\leqslant\min_k|f'(\tilde{z}_k)|$; $\{z_1, z_2, z_3\}$ is a triangle in C_{d+1} of a special triangulation in* [1] (see Figure 4). *If $\max_k|z_k-\tilde{z}_j|\leqslant\min\left\{1,\ \frac{N}{5M}\right\}=\sigma$ for some $\tilde{z}_j$, then $\{z_1, z_2, z_3\}$ is not labelled* (1, 3, 2).

Proof. Since $f(\tilde{z}_j)=0$, according to Taylor's formula,
$$f(z)=f'(\tilde{z}_j)(z-\tilde{z}_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}(z-\tilde{z}_j)^l,$$
and we obtain
$$\frac{f(z_2)-f(z_3)}{f(z_1)-f(z_3)}=\frac{f'(\tilde{z}_j)(z_2-z_3)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}[(z_2-\tilde{z}_j)^l-(z_3-\tilde{z}_j)^l]}{f'(\tilde{z}_j)(z_1-z_3)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}[(z_1-\tilde{z}_j)^l-(z_3-\tilde{z}_j)^l]}$$
$$=\frac{z_2-z_3}{z_1-z_3}\left[1+\frac{\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}\sum_{s=1}^{l}(z_2-\tilde{z}_j)^{l-s}(z_3-\tilde{z}_j)^{s-1}-\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde{z}_j)^{l-s}(z_3-\tilde{z}_j)^{s-1}}{f'(\tilde{z}_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde{z}_j)^{l-s}(z_3-\tilde{z}_j)^{s-1}}\right].$$

When $\max_k|z_k-\tilde{z}_j|\leqslant\min\left\{1,\ \frac{N}{5M}\right\}=\sigma$, we have
$$\left|\frac{\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l}\sum_{s=1}^{l}(z_2-\tilde{z}_j)^{l-s}(z_3-\tilde{z}_j)^{s-1}-\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde{z}_j)^{l-s}(z_3-\tilde{z}_j)^{s-1}}{f'(\tilde{z}_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde{z}_j)}{l!}\sum_{s=1}^{l}(z_1-\tilde{z}_j)^{l-s}(z_3-\tilde{z}_j)^{s-1}}\right|$$
$$\leqslant\frac{2\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{l!}l\sigma^{l-1}}{|f'(\tilde{z}_j)|-\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{l!}l\sigma^{l-1}}\leqslant\frac{2\sigma\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{(l-1)!}}{N-\sigma\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{(l-1)!}}=\frac{2\sigma M}{N-\sigma M}\leqslant\frac{1}{2},$$
so
$$\frac{\pi}{12}=\frac{\pi}{4}-\frac{\pi}{6}\leqslant\arg\frac{z_2-z_3}{z_1-z_3}-\frac{\pi}{6}\leqslant\arg\frac{f(z_2)-f(z_3)}{f(z_1)-f(z_3)}\leqslant\arg\frac{z_2-z_3}{z_1-z_3}+\frac{\pi}{6}\leqslant\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$$
(see Figure 1).

If $\{z_1, z_2, z_3\}$ is a triangle labelled (1, 3, 2), without loss of generality, we only have to show the case of Figure 2. Then it follows that

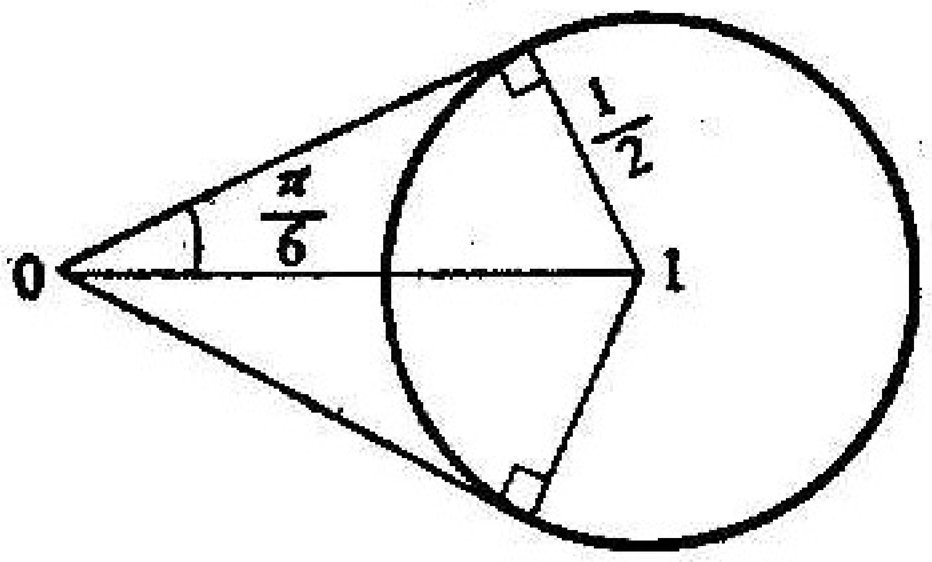

Fig. 1

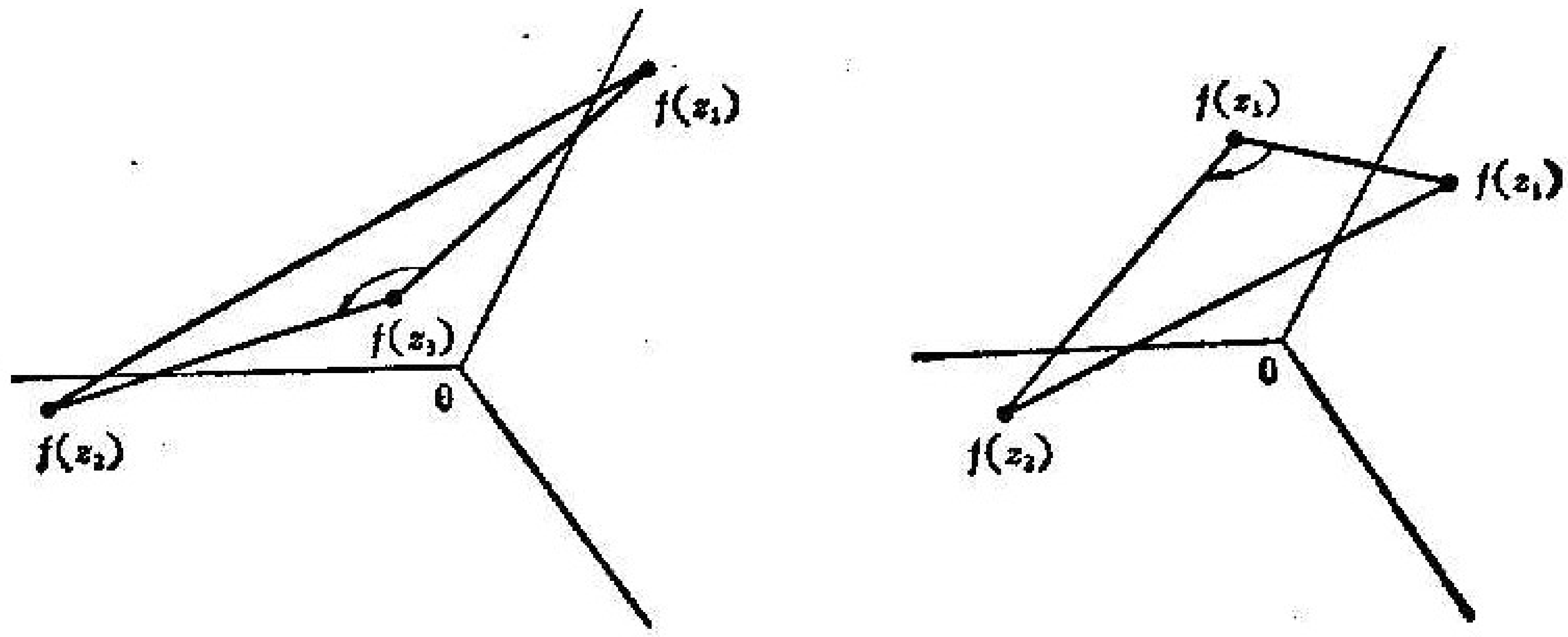

Fig. 2

$$\arg\frac{f(z_2)-f(z_3)}{f(z_1)-f(z_3)}>\frac{2\pi}{3}$$

or

$$\arg\frac{f(z_2)-f(z_3)}{f(z_1)-f(z_3)}<0.$$

This leads to contradiction and proves the lemma.

Lemma 1.3. *If $\tilde{z}_1, \cdots, \tilde{z}_n$ are the simple roots of $f(z)$, then there exists*

$$D\geqslant\log_2\frac{\left(1+\frac{3}{4}n\right)\sqrt{2}\,h}{\min\{1,\ N/5M\}}$$

such that when $d\geqslant D$, there is no triangle labelled $(1, 3, 2)$ *in C_d.*

Proof. Let $\{z_1, z_2, z_3\}$ be a triangle labelled $(1, 3, 2)$. Then there is some $\tilde{z}_j$, such that

$$|z_k-\tilde{z}_j|\leqslant|z_k-z_{k_0}|+|z_{k_0}-\tilde{z}_j|\leqslant\frac{\sqrt{2}\,h}{2^d}+\frac{3}{4}n\frac{\sqrt{2}\,h}{2^d}=\left(1+\frac{3}{4}n\right)\frac{\sqrt{2}\,h}{2^d},$$

where $|z_{k_0}-\tilde{z}_j|=\min\limits_k|z_k-\tilde{z}_j|$. Taking

$$D\geqslant\log_2\frac{(1+\frac{3}{4}n)\sqrt{2}\,h}{\min\{1,\ N/5M\}},$$

we obtain, for $d\geqslant D$,

$$\max_k|z_k-\tilde{z}_j|\leqslant\left(1+\frac{3}{4}n\right)\frac{\sqrt{2}\,h}{2^d}\leqslant\left(1+\frac{3}{4}n\right)\frac{\sqrt{2}\,h}{2^D}\leqslant\min\left\{1,\ \frac{N}{5M}\right\}.$$

By Lemma 1.2, $\{z_1, z_2, z_3\}$ is not a triangle labelled $(1, 3, 2)$. This leads to contradiction and the lemma is proved.

Theorem 1.1. *If $\tilde{z}_1, \cdots, \tilde{z}_n$ are the simple roots of $f(z)$, then for D in Lemma 1.3, when $d\geqslant D$, both d and $d_{jk}+d_{jk'}+d_{jk''}+d_{jk'''}$ are increasing where (z_{jk}, d_{jk}) belongs to a tetrahedron $\{(z_{jk}, d_{jk}), (z_{jk'}, d_{jk'}), (z_{jk''}, d_{jk''}), (z_{jk'''}, d_{jk'''})\}$, $k>k', k'', k'''$,* and *$d\leqslant d_{jk}, d_{jk'}, d_{jk''}, d_{jk'''}\leqslant d+1$.*

Proof. Obviously, Lemma 1.3 implies that d is increasing for $d\geqslant D$. To prove $d_{jk}+d_{jk'}+d_{jk''}+d_{jk'''}$ is increasing too, we consider the following cases.

In Figure 3, by means of the projection $\{z_1, z_2, z_3\}$ of the triangle $\{(z_1, d), (z_2, d+1), (z_3, d)\}$ and Lemma 1.3, we know the triangle is not labelled$(1, 3, 2)$.

In Figure 4, if $\{(z_1, d), (z_1, d+1), (z_2, d+1), (z_3, d+1)\}$ is a tetrahedron with a face labelled $(1, 2, 3)$ or $(1, 3, 2)$, then according to Lemma 1.3, $\{(z_1, d+1), (z_2, d+1), (z_3, d+1)\}$ must be a triangle labelled$(1, 2, 3)$.

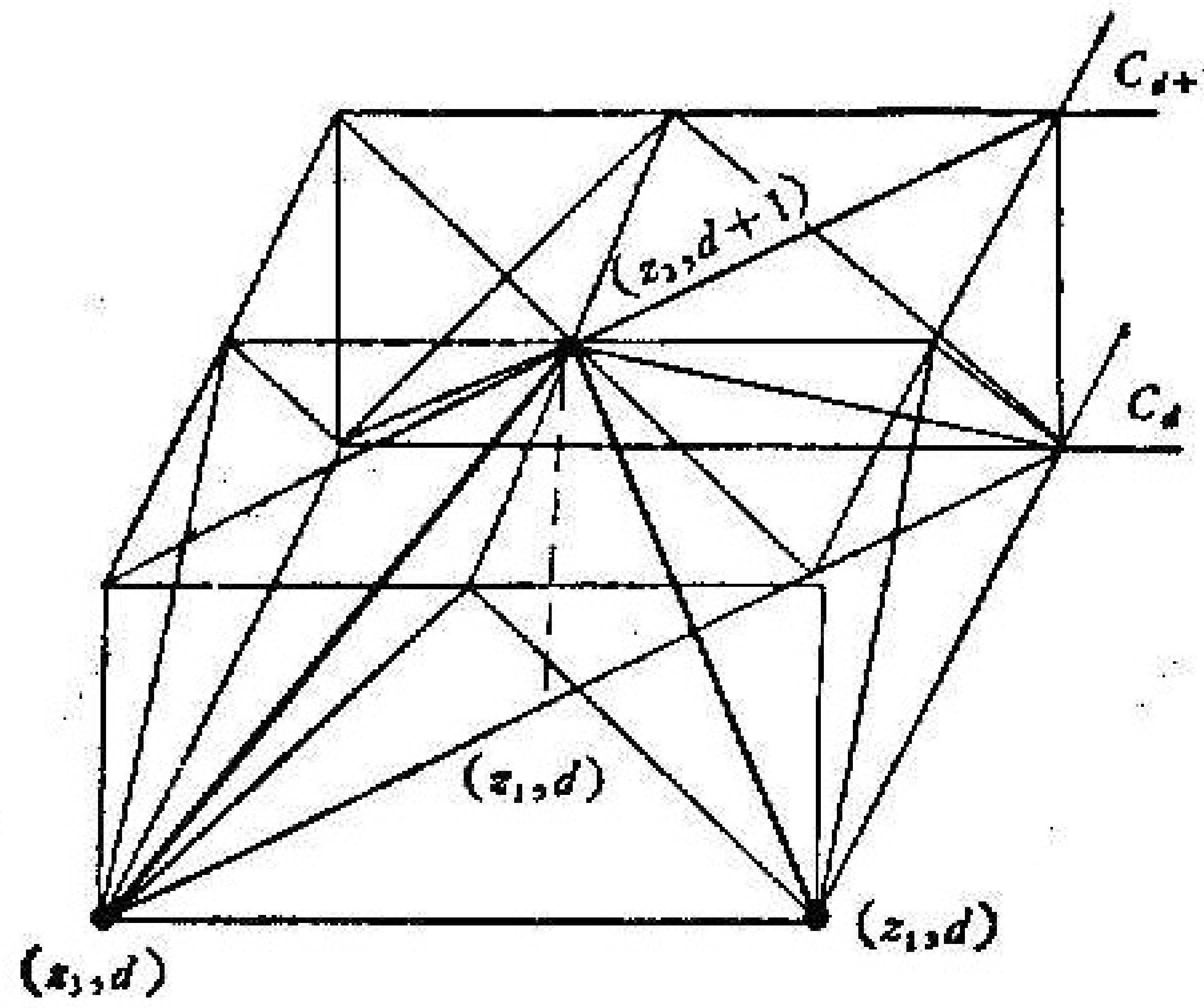

Fig. 3

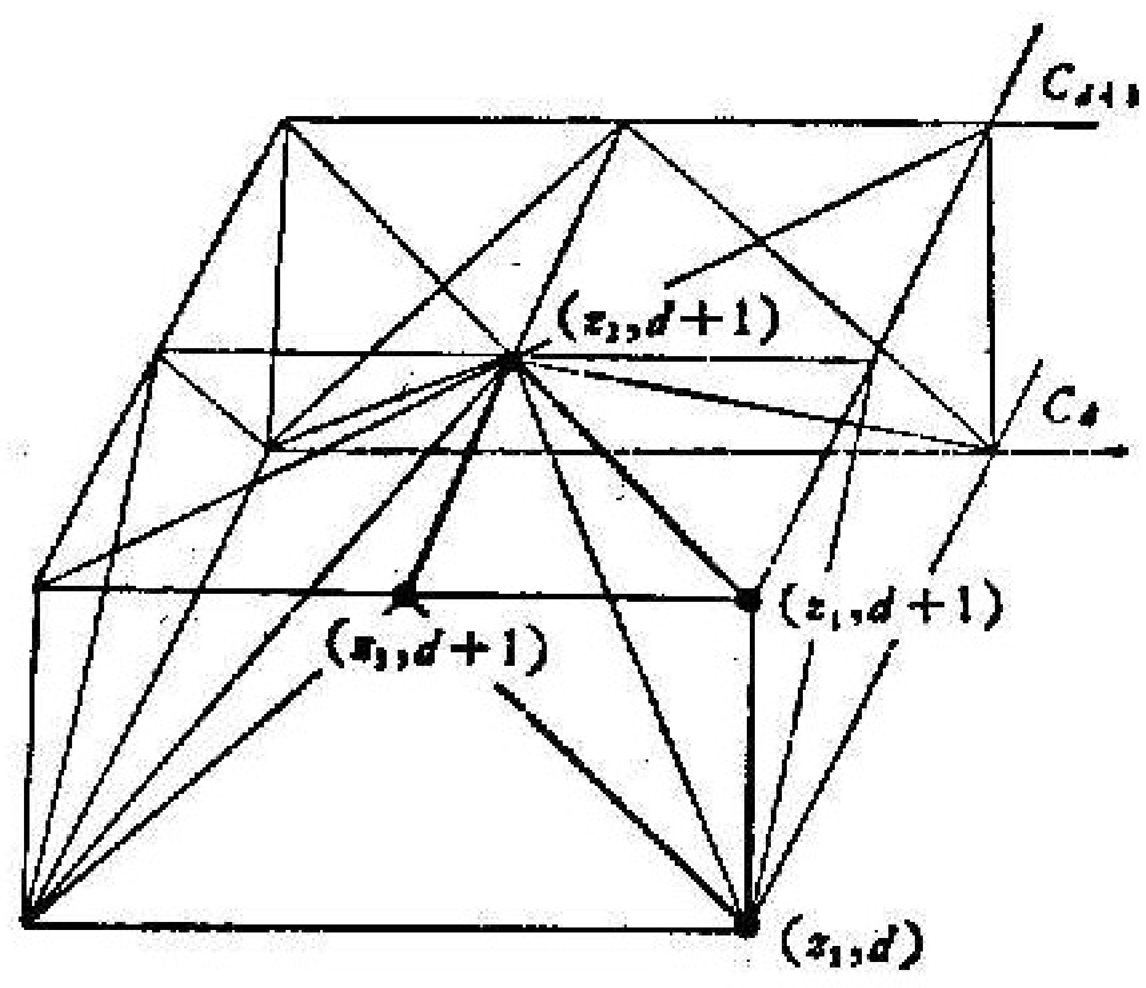

Fig. 4

Finally, by use of symmetry of the triangulation in Figure 4, we see from Figure 4 that $d_{jk}+d_{jk'}+d_{jk''}+d_{jk'''}$ is increasing.

2. The Expression of D in a_k

In Lemma 1.3, $D \geqslant \log_2 \dfrac{(1+\frac{3}{4}n)\sqrt{2}\,h}{\min\{1,\ N/5M\}}$. Since M can be expressed in terms of a_k, we need only to give the expression of N in a_k.

Let $f(z)=\prod_{l=1}^{n}(z-\tilde{z}_l)$, then $f'(\tilde{z}_j)=\prod_{l\neq j}(\tilde{z}_j-\tilde{z}_l)$. From the Vandermonde determinant, we obtain

$$\Delta=\left(\begin{vmatrix} n & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{vmatrix}\right)^{\frac{1}{2}}$$

$$=\left[\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tilde{z}_1 & \tilde{z}_2 & \cdots & \tilde{z}_n \\ \tilde{z}_1^2 & \tilde{z}_2^2 & \cdots & \tilde{z}_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{z}_1^{n-1} & \tilde{z}_2^{n-1} & \cdots & \tilde{z}_n^{n-1} \end{pmatrix}\begin{pmatrix} 1 & \tilde{z}_1 & \tilde{z}_1^2 & \cdots & \tilde{z}_1^{n-1} \\ 1 & \tilde{z}_2 & \tilde{z}_2^2 & \cdots & \tilde{z}_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \tilde{z}_n & \tilde{z}_n^2 & \cdots & \tilde{z}_n^{n-1} \end{pmatrix}\right]^{\frac{1}{2}}$$

$$=\Big|\prod_{i>l}(\tilde{z}_i-\tilde{z}_l)\Big| \leqslant n^2 R \prod_{l\neq j}|\tilde{z}_j-\tilde{z}_l|,$$

where $s_k=\sum_{l=1}^{n}\tilde{z}_l^k$. So we have

$$|f'(\tilde{z}_j)|=\prod_{l\neq j}|\tilde{z}_j-\tilde{z}_l| \geqslant \frac{\Delta}{n^2R}>0$$

and

$$\min_k |f'(\tilde{z}_k)| \geqslant \frac{\Delta}{n^2R}.$$

Take $N=\dfrac{\Delta}{n^2R}$. Because the determinant above is a symmetric polynomial in $\tilde{z}_1, \cdots, \tilde{z}_n$, it can be expressed as a polynomial in elementary symmetric polynomials $\sum_{i_1<\cdots<i_k}\tilde{z}_{i_1}\cdots\tilde{z}_{i_k}=(-1)^k a_k$ in lexicographic ordering (see [2]).

3. Just One Triangle Labelled (1, 2, 3) in $U(\tilde{z}^j, L)$

In Figure 5, Q_m is composed of $4m^2$ small squares, $\tilde{z}\in$ Square $ABCO$.

Lemma 3.1. *For* $m>\max\left\{3,\ \frac{n}{\pi}+2\right\}$ *and the edge* $(z_1, z_2)\in\partial Q_m$, *there holds the inequality*

$$\frac{n}{4m}<\arg\frac{(z_2-\tilde{z})^n}{(z_1-\tilde{z})^n}<\frac{n}{m-2}\leqslant\pi.$$

Proof. Consider the edge (z_1, z_2) on the straight line EF. By the formula of triangle area

$$\frac{1}{2}\sqrt{(m-\varepsilon_1)^2+(l-\varepsilon_2)^2}\sqrt{(m-\varepsilon_1)^2+(l+1-\varepsilon_2)^2}\sin\theta=\frac{1}{2}(m-\varepsilon_1)\cdot 1,$$

$-m\leqslant l\leqslant m-1$, we obtain

$$\begin{aligned}\frac{1}{4m}&<\frac{m-1}{m^2+(m+1)^2}\leqslant\frac{m-\varepsilon_1}{\sqrt{(m-\varepsilon_1)^2+(l-\varepsilon_2)^2}\sqrt{(m-\varepsilon_1)^2+(l+1-\varepsilon_2)^2}}\\
&=\sin\theta<\theta<\operatorname{tg}\theta=\sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\\
&=\sqrt{\frac{(m-\varepsilon_1)^2}{[(m-\varepsilon_1)^2+(l-\varepsilon_2)^2][(m-\varepsilon_1)^2+(l+1-\varepsilon_2)^2]-(m-\varepsilon_1)^2}}\\
&=\frac{m-\varepsilon_1}{(m-\varepsilon_1)^2+(l+1-\varepsilon_2)(l-\varepsilon_2)}\leqslant\frac{m}{(m-1)^2-1/4}\\
&=\frac{4m}{4(m-1)^2-1}<\frac{1}{m-2}.\end{aligned}$$

Similarly, for the edge (z_1, z_2) on the straight line GH, we obtain

$$\begin{aligned}\frac{1}{4m}&<\frac{m}{2(m+1)^2}\leqslant\frac{m+\varepsilon_1}{\sqrt{(m+\varepsilon_1)^2+(l-\varepsilon_2)^2}\sqrt{(m+\varepsilon_1)^2+(l+1-\varepsilon_2)^2}}\\
&=\sin\theta<\theta<\operatorname{tg}\theta=\sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\\
&=\sqrt{\frac{(m+\varepsilon_1)^2}{[(m+\varepsilon_1)^2+(l-\varepsilon_2)^2][(m+\varepsilon_1)^2+(l+1-\varepsilon_2)^2]-(m+\varepsilon_1)^2}}\\
&=\frac{m+\varepsilon_1}{(m+\varepsilon_1)^2+(l+1-\varepsilon_2)(l-\varepsilon_2)}<\frac{m+1}{m^2-1/4}\\
&=\frac{4(m+1)}{4m^2-1}<\frac{1}{m-1},\qquad -m\leqslant l\leqslant m-1.\end{aligned}$$

Therefore

$$\frac{1}{4m}<\theta=\arg\frac{z_2-\tilde{z}}{z_1-\tilde{z}}<\frac{1}{m-2},$$

$$\frac{n}{4m}<\arg\frac{(z_2-\tilde{z})^n}{(z_1-\tilde{z})^n}<\frac{n}{m-2}\leqslant\pi.$$

Lemma 3.2. *Suppose* $\tilde{z}_j$ *is a simple root of* $f(z)$ and $\tilde{z}=\tilde{z}_j$ *in Lemma* 3.1. *For the*

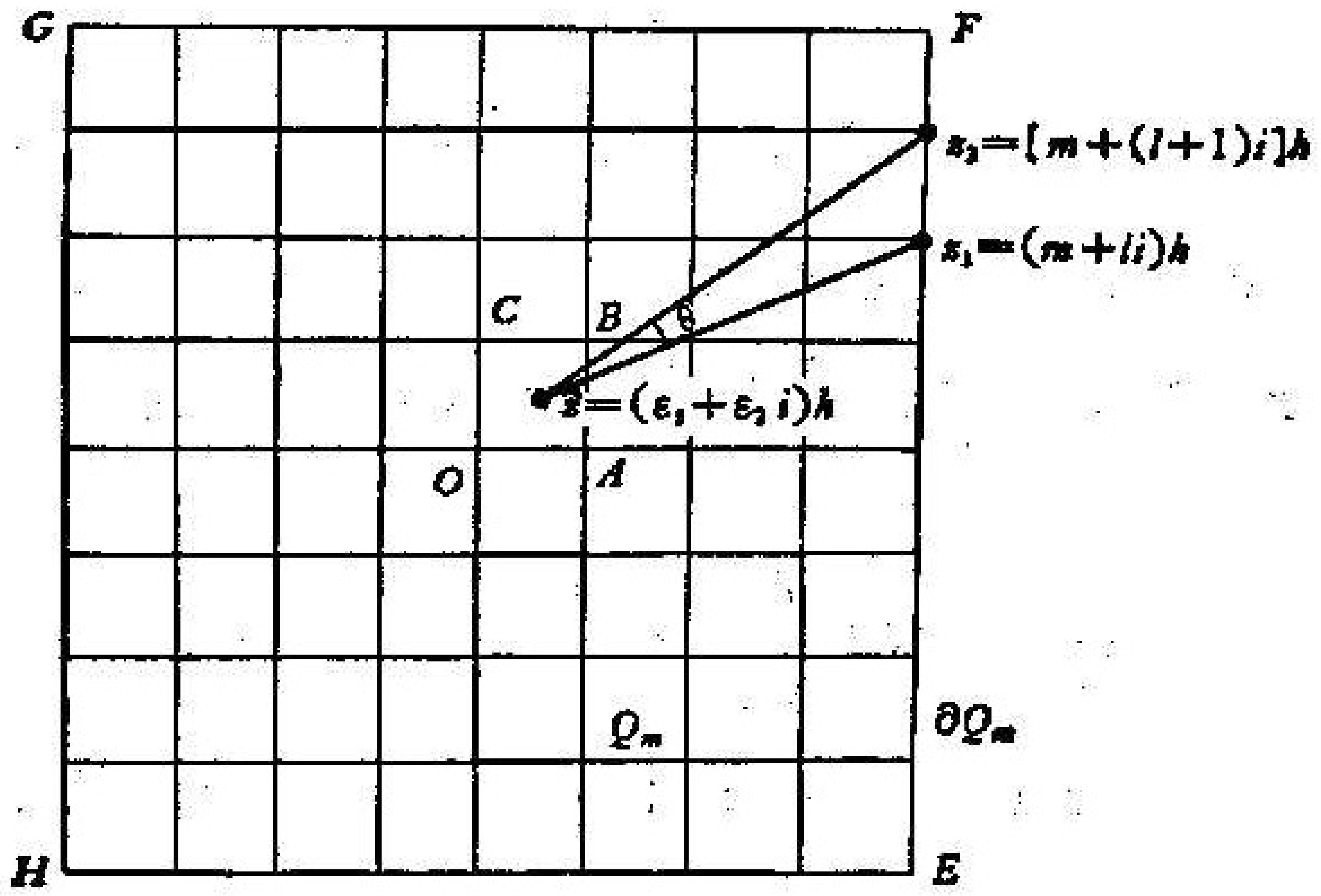

Fig. 5

edge $(z_1, z_2)\in\partial Q_m$, $m\geqslant 5$, *and* $\max\limits_k|z_k-\tilde{z}_j|<\min\left\{1, \dfrac{N}{(16m+1)M_1}\right\}=\sigma_1$, $M_1=1+\sum\limits_{l=2}^{n}\dfrac{\varphi^{(l)}(R)}{l!}$, *we have*

$$\frac{1}{20m}<\arg\frac{f(z_2)}{f(z_1)}<\frac{1}{m-2}+\frac{\pi}{16m}<\frac{\pi}{6}.$$

Proof. By means of Taylor's formula,

$$\arg\frac{f(z_2)}{f(z_1)}=\arg\frac{f'(\tilde{z}_j)(z_2-\tilde{z}_j)+\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_2-\tilde{z}_j)^l}{f'(\tilde{z}_j)(z_1-\tilde{z}_j)+\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_1-\tilde{z}_j)^l}$$

$$=\arg\frac{z_2-\tilde{z}_j}{z_1-\tilde{z}_j}\left[1+\frac{\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_2-\tilde{z}_j)^{l-1}-\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_1-\tilde{z}_j)^{l-1}}{f'(\tilde{z}_j)+\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_1-\tilde{z}_j)^{l-1}}\right].$$

Let $M_1=1+\sum\limits_{l=2}^{n}\dfrac{\varphi^{(l)}(R)}{l!}$, $\sigma_1=\min\left\{1, \dfrac{N}{(16m+1)M_1}\right\}$, $\max\limits_k|z_k-\tilde{z}_j|<\sigma_1$, then

$$\left|\frac{\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_2-\tilde{z}_j)^{l-1}-\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_1-\tilde{z}_j)^{l-1}}{f'(\tilde{z}_j)+\sum\limits_{l=2}^{n}\dfrac{f^{(l)}(\tilde{z}_j)}{l!}(z_1-\tilde{z}_j)^{l-1}}\right|$$

$$\leqslant\frac{2\sum\limits_{l=2}^{n}\dfrac{\varphi^{(l)}(R)}{l!}\sigma_1^{l-1}}{|f'(\tilde{z}_j)|-\sum\limits_{l=2}^{n}\dfrac{\varphi^{(l)}(R)}{l!}\sigma_1^{l-1}}\leqslant\frac{2\sigma_1\sum\limits_{l=2}^{n}\dfrac{\varphi^{(l)}(R)}{l!}}{|f'(\tilde{z}_j)|-\sigma_1\sum\limits_{l=2}^{n}\dfrac{\varphi^{(l)}(R)}{l!}}\leqslant\frac{2\sigma_1 M_1}{N-\sigma_1 M_1}<\frac{1}{8m}.$$

As $|\arg(1+w)|\leqslant\dfrac{\pi}{2}|w|$ (when $|w|<1$) and by Lemma 3.1, we obtain

$$\frac{1}{20m}<\frac{1}{4m}-\frac{\pi}{2}\cdot\frac{1}{8m}<\arg\frac{f(z_2)}{f(z_1)}<\frac{1}{m-2}+\frac{\pi}{2}\cdot\frac{1}{8m}=\frac{1}{m-2}+\frac{\pi}{16m}<\frac{\pi}{6}.$$

Similar to Proposition 4.1 in [1], we have

Corollary 3.1. If $Q_m\subset U(\tilde{z}_j,\sigma_1)=\{z:|z-\tilde{z}_j|<\sigma_1\}$, then there is no edges labelled (1, 3), (3, 2) or (2, 1) on ∂Q_m.

Proof. This follows immediately from the inequality in Lemma 3.2.

Lemma 3.3. *Let* $\sigma_2=\min\left\{1,\ \frac{N}{3M_1}\right\}$; $\tilde z_1, \cdots, \tilde z_n$ *are the simple roots of* $f(z)$. *Then the labelled region of* $f(z)$ *in* $U(\tilde z_j, \sigma_2)=\{z: |z-\tilde z_j|<\sigma_2\}$ *is such as shown in Figure* 6.

Proof. By Taylor's formula,

$$f(z)=f'(\tilde z_j)(z-\tilde z_j)+\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}(z-\tilde z_j)^l$$

$$=f'(\tilde z_j)(z-\tilde z_j)\left[1+\frac{1}{f'(\tilde z_j)}\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}(z-\tilde z_j)^{l-1}\right]$$

and when $|z-\tilde z_j|<\min\left\{1,\ \frac{N}{3M_1}\right\}=\sigma_2$, we have

$$\left|\frac{1}{f'(\tilde z_j)}\sum_{l=2}^{n}\frac{f^{(l)}(\tilde z_j)}{l!}(z-\tilde z_j)^{l-1}\right|\leqslant\frac{\sigma_2\sum_{l=2}^{n}\frac{\varphi^{(l)}(R)}{l!}}{N}\leqslant\frac{\sigma_2 M_1}{N}<\frac{1}{3}.$$

So,
$$\left|\arg\frac{f(z)}{f'(\tilde z_j)(z-\tilde z_j)}\right|\leqslant\frac{\pi}{2}\cdot\frac{1}{3}=\frac{\pi}{6}.$$

This implies that the labelled region of $f(z)$ in $U(\tilde z_j, \sigma_2)$ is such as shown by Figure 6, where $f'(\tilde z_j)=re^{i\eta}$.

Corollary 3.2. In Q_m, $\tilde z=\tilde z_j$ is a simple root of $f(z)$. When $Q_m\subset U(\tilde z_j, \sigma_1)$, $m\geqslant 5$, there is just one edge labelled (1, 2) and no edge labelled (2, 1) on ∂Q_m.

Proof. Since $\sigma_1=\min\left\{1,\ \frac{N}{(16m+1)M_1}\right\}\leqslant\min\left\{1,\ \frac{N}{3M_1}\right\}=\sigma_2$, by Corollary 3.1, there is no edge labelled (1, 3), (3, 2) or (2, 1) on ∂Q_m. From Figure 6 we see that the label of each vertex should be such as shown in Figure 7. So, there is just one edge labelled (1, 2) and no edge labelled (2, 1) on ∂Q_m.

Theorem 3.1. *Let* $\tilde z_1, \cdots, \tilde z_n$ *be the simple roots of* $f(z)$; $0<L\leqslant 1/2\min\limits_{k\neq l}|\tilde z_k-\tilde z_l|$. *When* $d\geqslant D\geqslant\max\left\{\log_2\frac{(m+1)\sqrt{2}h}{\min|L,\sigma_1\}},\ \log_2\frac{(1+\frac{3}{4}n)\sqrt{2}h}{\min\{1,\ N/5M\}}\right\}$ *there is just one triangle labelled* (1, 2, 3) *in* $U(\tilde z_j, L)\subset C_d$ *and no triangle labelled* (1, 3, 2), *where*

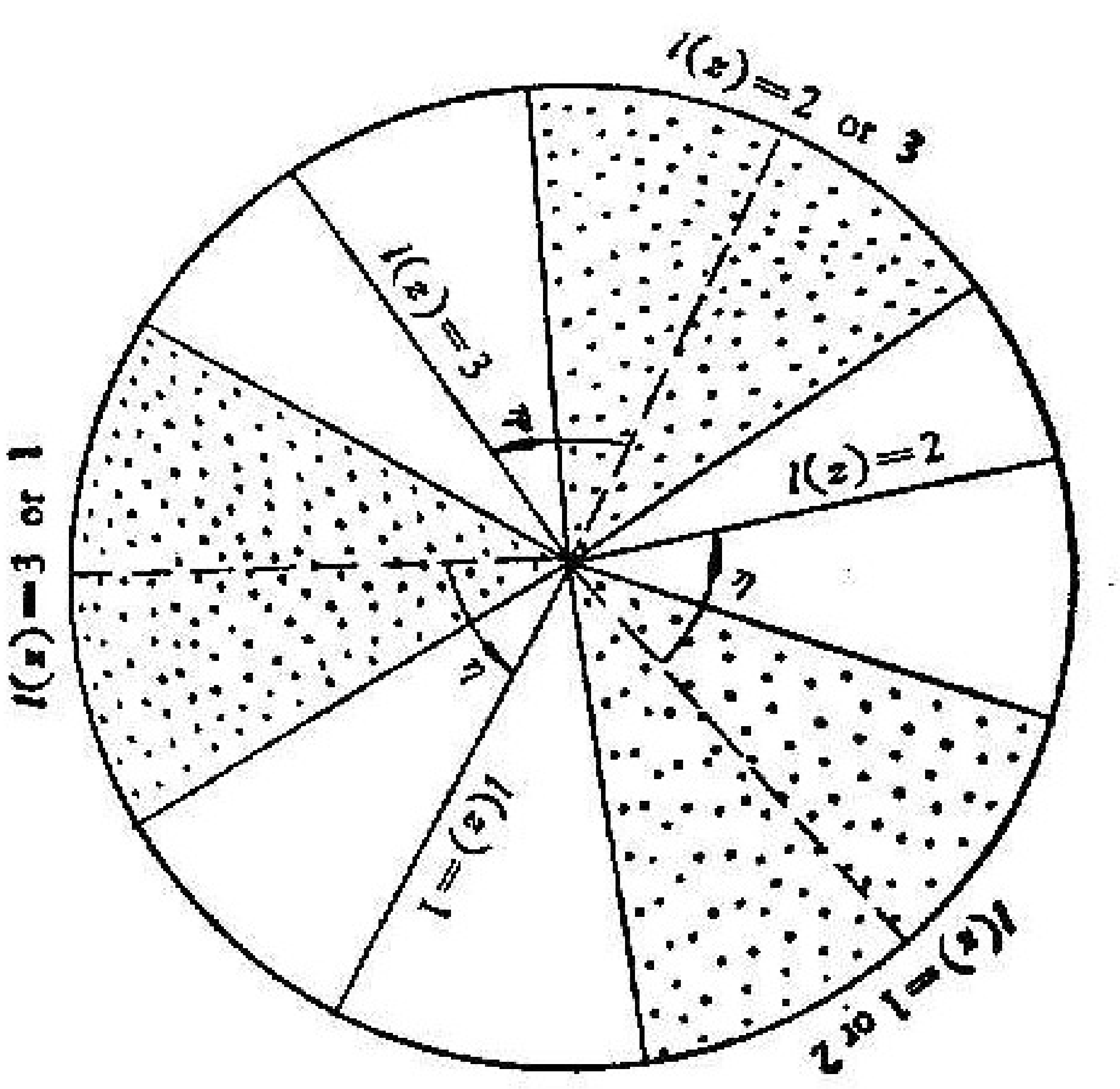

Fig. 6

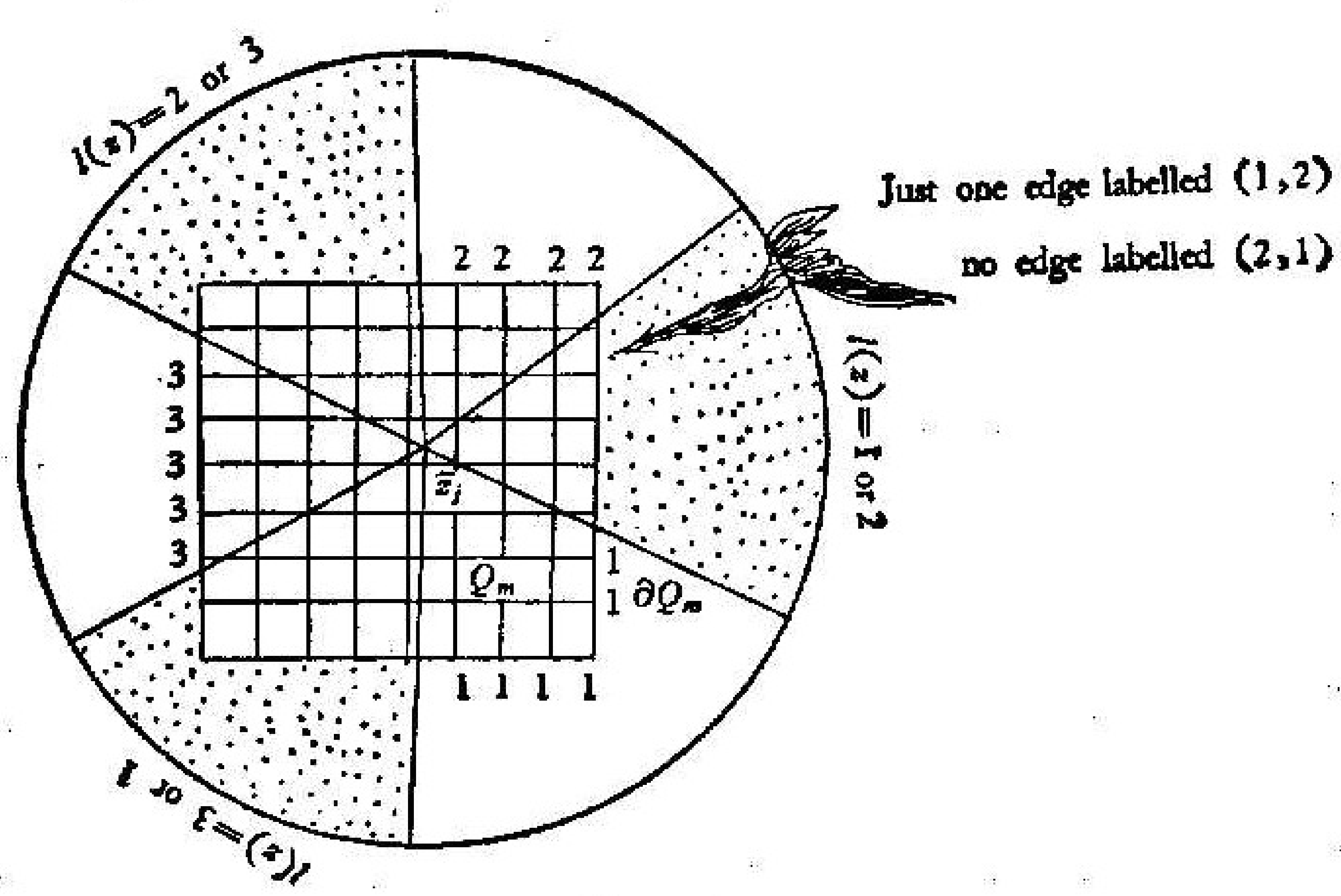

Fig. 7

$$m \geqslant \max\left\{5,\ \left(1+\frac{3}{4}n\right)\sqrt{2}+1\right\}.$$

Proof. Take $D \geqslant \log_2 \frac{(m+1)\sqrt{2}h}{\min\{L,\ \sigma_1\}}$. Then when $d \geqslant D$, we have

$$(m+1)\frac{\sqrt{2}h}{2^d} \leqslant (m+1)\frac{\sqrt{2}h}{2^D} < \min\{L,\ \sigma_1\}$$

and
$$Q_m \subset U(\tilde{z}_j,\ \min\{L,\ \sigma_1\}) = U(\tilde{z}_j,\ L) \cap U(\tilde{z}_j,\ \sigma_1).$$

Therefore there is just one edge labelled (1, 2) and no edge labelled (2, 1) on ∂Q_m, where $m \geqslant 5$.

Fix m such that

$$m \geqslant \left(1+\frac{3}{4}n\right)\sqrt{2}+1.$$

For the triangle $\{z_1,\ z_2,\ z_3\}$ labelled (1, 2, 3) or (1, 3, 2) in C_d, using the proof of Lemma 1.3 we have

$$\max_k |z_k - \tilde{z}_j| \leqslant \left(1+\frac{3}{4}n\right)\frac{\sqrt{2}h}{2^d} < (m-1)\frac{h}{2^d} \leqslant (m-1)\frac{h}{2^D} < L.$$

Then $\{z_1,\ z_2,\ z_3\} \subset Q_m \subset U(\tilde{z}_j,\ L)$.

By Lemma 1.3, when

$$d \geqslant D \geqslant \log_2 \frac{(1+\frac{3}{4}n)\sqrt{2}h}{\min\{1,\ N/5M\}},$$

there is no triangle labelled (1, 3, 2).

Finally, by means of Combinatorial Stokes' Theorem (see [3]), when

$$d \geqslant D \geqslant \max\left\{\log_2 \frac{(m+1)\sqrt{2}h}{\min\{L,\ \sigma_1\}},\ \log_2 \frac{(1+\frac{3}{4}n)\sqrt{2}h}{\min\{1,\ N/5M\}}\right\}$$

where
$$m \geqslant \max\left\{5,\ \left(1+\frac{3}{4}n\right)\sqrt{2}+1\right\},$$

there is just one triangle labelled (1, 2, 3) in $U(\tilde{z}_j,\ L)$ and no triangle labelled (1, 3, 2).

Remark. Similar to Section 2, we write L in a_k. Since

$$\Delta = \prod_{i>l} |\tilde{z}_i - \tilde{z}_l| \leqslant n^2 R \min_{k \neq l} |\tilde{z}_k - \tilde{z}_l|,$$

$$\frac{1}{2}\min_{k \neq l} |\tilde{z}_k - \tilde{z}_l| \geqslant \frac{\Delta}{2n^2 R},$$

we take
$$L = \frac{\Delta}{2n^2 R}.$$

The authors are most grateful to Professor W. C. Hsiang and Professor H. W. Kuhn for their direction and help.

References

[1] H. W. Kuhn, Finding Roots of Polynomials by Pivoting, *Fixed Points Algorithms and Applications*, Ed. Stepan Karamardian (1977), 11—39.

[2] Nathan Jacobson, Basic Algebra 1, Freeman, 1974.

[3] H. W. Kuhn, A New Proof of the Fundamental Theorem of Algebra, *Mathematical Programming Study* 1, Ed. M. Balinski, North-Holland, 1974, 148—158.