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TWO-GRID CHARACTERISTIC FINITE VOLUME METHODS FOR NONLINEAR PARABOLIC PROBLEMS*

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Abstract

In this work, two-grid characteristic finite volume schemes for the nonlinear parabolic problem are considered. In our algorithms, the diffusion term is discretized by the finite volume method, while the temporal differentiation and advection terms are treated by the characteristic scheme. Under some conditions about the coefficients and exact solution, optimal error estimates for the numerical solution are obtained. Furthermore, the two-grid characteristic finite volume methods involve solving a nonlinear equation on coarse mesh with mesh size H, a large linear problem for the Oseen two-grid characteristic finite volume method on a fine mesh with mesh size $h = \mathcal{O}(H^2)$ or a large linear problem for the Newton two-grid characteristic finite volume method on a fine mesh with mesh size $h = \mathcal{O}(|\log h|^{1/2}H^3)$. These methods we studied provide the same convergence rate as that of the characteristic finite volume method, which involves solving one large nonlinear problem on a fine mesh with mesh size h. Some numerical results are presented to demonstrate the efficiency of the proposed methods.

Mathematics subject classification: 35Q55, 65N30, 76D05.
Key words: Two-grid, Characteristic finite volume method, Nonlinear parabolic problem, Error estimate, Numerical example.

1. Introduction

Many processes in science and engineering are described by the parabolic equations, for instance, the processes of fluid dynamics, hydrology and environmental protection [20, 25]. There have been extensive works devoted to linear parabolic problems see, e.g., the monographs [30]. For nonlinear cases, we mention only [9, 26] and the references therein.

In this paper, we consider the following nonlinear parabolic problem in $\mathbb{R}^2$:

$$\begin{cases} u_t + \nabla\cdot(a(u)\nabla u) + \mathbf{b}(u)\nabla u = f(u), & \text{in } \Omega\times(0,T], \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T], \\ u(\cdot,0) = u_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where Ω is a bounded convex polygonal domain with a sufficiently smooth boundary $\partial\Omega$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^T$, and $\mathbf{b}(u) = (b_1(u), b_2(u))^T$ is a vector function. We define a bounded set on $\mathbb{R}^2$ as

$$G = \{u : |u| \le K_0\}, \tag{1.2}$$

where K_0 is a positive constant.

Supposing the coefficients of problem (1.1) satisfy the following conditions:

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$(C_1):$ $a(u)$ and $f(u)$ are Lipschitz continuous with respect to u, i.e.

$$|g(u)-g(v)| \le L|u-v|, \qquad \text{for } \forall\ u,v \in G, \tag{1.3a}$$

where L is a Lipschitz constant related to K_0, $g(u)$ can take $a(u)$ or $f(u)$.
$(C_2):$ $a(u)$ is a bounded smooth function with positive upper and lower bounds,

$$0 < a_* \le a(u) \le a^*, \quad \text{for } \forall\ u \in G. \tag{1.3b}$$

$(C_3):$ $f(u)$ is a given real-valued function on Ω and there is a constant M such that

$$|f'(u)| + |f''(u)| \le M, \qquad \text{for } \forall\ u \in G, \tag{1.3c}$$

where $f'(u) = \frac{df(u)}{du}$. Under the conditions above, problem (1.1) admits a unique solution in a certain Sobolev space (see [30]).

Finite volume method (FVM) as one of numerical discretization techniques has been widely employed to solve the fluid dynamics problems in recent years (see [12, 23] and the references therein). It is developed as an attempt to use finite element ideas in the finite difference setting. The basic idea is to approximate the discrete fluxes of a partial differential equation using the finite element procedure based on volumes or control volumes, so FVM is also called box scheme, general difference method et al. (see [1,3,19]). Finite volume method has many advantages that belong to finite difference method or finite element method, such as, it is easy to set up and implement, conserve mass locally and it also can treat the complicated geometry and general boundary conditions flexibility. However, the analysis of FVM lags far behind that of finite element and finite difference methods, we can refer to the literature [10, 11, 31] for more recent developments about the finite volume method.

The modified method of characteristic (MMOC) was first proposed by Douglas and Russell for the convection-diffusion equations in [8]. After then, a lot of works have been reported about this method. For instance, Russell considered the nonlinear coupled systems in [27], Süli studied the Navier-Stokes equations in [29]. The MMOC is based on the approximation of the material derivative term, that is, the time derivative term plus the convection term, and this scheme works well for convection dominant problem (see [35] and the reference therein).

On the other hand, two-grid method is an efficient numerical scheme for partial differential equations based on two spaces with different mesh sizes. This kind of discretization technique for linear and nonlinear elliptic PDEs was first introduced by Xu in [32, 33]. After then, two-grid method has been studied by many researchers, for example, Dawson et al. considered the nonlinear parabolic equations by using the finite element or finite difference methods in [6, 7], respectively. Marion and Xu [24] applied it to the evolution equations. For the Navier-Stokes equations, we can refer to [15–18, 22]. Recently, Bi and Ginting in [2] combined the two-grid method and the finite volume method for linear and nonlinear elliptic problems.

In this paper, we devote ourselves to the study of two-grid characteristic finite volume method (CFVM) for nonlinear parabolic problem. By introducing an elliptic projection, optimal error estimates of numerical solution are established. Another important novel ingredient of this work is the convergence analysis of the approximate solution in two-grid schemes. We prove that the initial approximation u_H^n of the nonlinear problem is determined on the coarse mesh. Then the fine mesh approximation u_h^{no} or u_h^{nn} is obtained by solving a large linear problem for the Oseen two-grid CFVM on a fine mesh with mesh size $h = \mathcal{O}(H^2)$ or a large linear problem for the Newton two-grid CFVM on a fine mesh with mesh size $h = \mathcal{O}(|\log h|^{1/2}H^3)$, respectively.

For the usual characteristic finite volume approximation u_h^n, which involves solving one large nonlinear parabolic problem on a fine mesh with mesh size h, we provide the following error estimate:

$$\|u - u_h^n\|_1 \leq C(\Delta t + h). \tag{1.4}$$

Here and below, the letter C denotes a positive constant, independent of mesh parameter h and time step Δt, and it may stand for different values at its different places. After that, we obtain that the Oseen two-grid characteristic finite volume solution u_h^{no} is of the following error estimate:

$$\|u - u_h^{no}\|_1 \leq C(\Delta t + h + H^2). \tag{1.5}$$

Finally, we show that the Newton two-grid characteristic finite volume solution u_h^{nn} is of the following error estimate:

$$\|u - u_h^{nn}\|_1 \leq C(\Delta t + h + |\log h|^{1/2} H^3). \tag{1.6}$$

Hence, if we choose H such that $h = \mathcal{O}(H^2)$ for the Oseen two-grid characteristic finite volume approximation or $h = \mathcal{O}(|\log h|^{1/2} H^3)$ for the Newton two-grid characteristic finite volume approximation, then the methods we studied are of the same convergence order as that of the usual CFVM. However, It turns out that our approach is simpler than the CFVM.

2. Preliminaries

In this section, we describe some notations and results which will be frequently used in this article. Standard notations are used for the Sobolev spaces $W^{s,p}(\Omega)$ with the norm $\|\cdot\|_{s,p,\Omega}$ and the semi-norms $|\cdot|_{s,p,\Omega}$. Denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and skip the index $p = 2$ for simplicity. For all $T > 0$ and integer number $n \geq 0$, define

$$H^n(0,T;W^{s,p}(\Omega)) = \Big\{ v \in W^{s,p}(\Omega); \sum_{0\leq i\leq n} \int_0^T (\frac{d^i}{dt^i}\|v\|_{s,p,\Omega})^2 dt < \infty \Big\},$$

and the corresponding norm of $H^n(0,T;W^{s,p}(\Omega))$ is denoted by

$$\|v\|_{H^n(0,T;W^{s,p}(\Omega))} = \sum_{0\leq i\leq n} \Big(\int_0^T (\frac{d^i}{dt^i}\|v\|_{s,p,\Omega})^2 \Big)^{\frac{1}{2}}.$$

Especially, when $n = 0$, we denote the norm as

$$\|v\|_{L^2(0,T;W^{s,p}(\Omega))} = \Big(\int_0^T \|v\|_{s,p,\Omega}^2 dt \Big)^{\frac{1}{2}}.$$

Let

$$L^\infty(0,T;W^{s,p}(\Omega)) = \Big\{ v \in W^{s,p}(\Omega); \text{ess} \sup_{0\leq t\leq T} \|v\|_{s,p,\Omega} < \infty \Big\},$$

with the corresponding norm

$$\|v\|_{L^\infty(0,T;W^{s,p}(\Omega))} = ess \sup_{0\leq t\leq T} \|v\|_{s,p,\Omega}.$$

Let T_h $(h > 0)$ denote a regular partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of triangulations K, h_k=diam(K), $h = \max_{K\in T_h} h_K$. All elements of T_h will be numbered so that $T_h = \{K_i\}_{i\in I}$, where $I \subset Z^+ = \{0, 1, 2, \ldots\}$ such that $\overline{\Omega} = \cup_{K_i\in T_h} K_i$, $\mathcal{N}_h$ denotes the set of all nodes T_h.

Based on the partition T_h, we introduce the corresponding dual partition T_h^*. Here, we choose the circum-center Q of a element $K \in T_h$, and the midpoints M on the edges of K, then connect Q to M by straight line. For an arbitrary vertex $x_i \in K$, let V_i be the polygonal which is called control volume. Then, we have $\overline{\Omega} = \cup_{x_i\in\mathcal{N}_h} V_i$, the dual mesh T_h^* is the set of these control volumes. We call the control volume mesh T_h^* is regular, i.e., there exists a positive constant C such that

$$C^{-1}h^2 \le \text{meas}(V_i) \le Ch^2, \qquad \forall\ V_i \in T_h^*.$$

Introduce a Lagrange interpolation operator I_h from $H^2(\Omega)$ into $H_0^1(\Omega)$, such that

$$\|u - I_h u\|_i \le Ch^{2-i}\|u\|_2, \qquad i = 0, 1,\ \forall\ u \in H^2(\Omega). \tag{2.1}$$

Let trial function space $U_h \subset H_0^1(\Omega)$ with basis functions $\{\phi_i(x)\}$ be a linear space based on T_h and the test function space $V_h \subset L^2(\Omega)$ be a piecewise constant space on the dual partition T_h^*, whose characteristic functions $\{\phi_i^*(x)\}$ are defined by

$$\phi_i^*(x) = \begin{cases} 1, & x \in V_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let I_h^* denote an interpolation operator from $H_0^1(\Omega)$ to V_h satisfying

$$I_h^* v = \sum_{x_i\in\mathcal{N}_h} v(x_i)\phi_i^*(x).$$

Set

$$\psi(x,t) = \sqrt{1 + |\mathbf{b(u)}|^2}, \quad \text{with} \quad |\mathbf{b(u)}|^2 = b_1(u)^2 + b_2(u)^2.$$

If we denote the characteristic direction corresponding to the hyperbolic part of (1.1), $u_t + \mathbf{b}(u)\nabla u$, by τ, then

$$\frac{\partial}{\partial\tau} = \frac{1}{\psi(x,t)}\frac{\partial}{\partial t} + \frac{1}{\psi(x,t)}\mathbf{b(u)}\cdot\nabla.$$

With this definition, we write (1.1) in the following equivalent form

$$\begin{cases} \psi(x,t)\frac{\partial u}{\partial\tau} + \nabla\cdot(a(u)\nabla u) = f(u), & \text{in } \Omega\times(0,T], \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T], \\ u(\cdot,0) = u_0, & \text{in } \Omega. \end{cases} \tag{2.2}$$

The weak form of characteristic finite volume method for problem (2.2) reads as

$$\Big(\psi(x,t)\frac{\partial u}{\partial\tau}, I_h^* v\Big) + a(u,u,I_h^* v) = (f(u), I_h^* v), \quad \forall\ v \in H_0^1(\Omega), \tag{2.3}$$

where $a(\cdot,\cdot,I_h^*\cdot)$ is defined by

$$a(w,u,I_h^* v) = \int_{\partial\Omega} (a(w)\nabla u)\cdot\mathbf{n}I_h^* v ds, \quad \forall\ w,u,v \in H_0^1(\Omega).$$

Here, $\mathbf{n}$ is the outside normal of the boundary $\partial\Omega$. Furthermore, we assume that, for $\forall (x,t) \in \Omega \times (0,T]$, the solution u of problem (1.1) satisfies the following regularities:

$$(C_4):\ u, u_t \in L^2(0,T;H^3(\Omega)), \quad \frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)), \quad u \in L^\infty(0,T;W^{1,\infty}(\Omega)).$$

Now, we consider a time step $\Delta t = T/N$ and approximate the solution at $t^n = n\Delta t, n = 1, ..., N$. The characteristic derivative can be approximated in the following way at $t = t^n$

$$\Big(\psi(x,t)\frac{\partial u}{\partial \tau}\Big)^n \approx \psi(x,t)\frac{u(x,t^n) - u(\overline{x}, t^{n-1})}{\sqrt{(x-\overline{x})^2 + \Delta t^2}} = \frac{u^n - \overline{u}^{n-1}}{\Delta t}.$$

Namely, a backtracking algorithm is used to approximate the characteristic derivative. $\overline{x} = x - u(x,t^n)\Delta t$ is the foot (at time level $t = t^{n-1}$) of the characteristic corresponding to x at the head (at time level $t = t^n$).

For any $v_h \in V_h$, the modified method of characteristic finite volume for problem (1.1) at $t = t^n$ reads as: Find $u_h^n \in U_h$ with time step Δt, such that

$$\Big(\frac{u_h^n - \overline{u}_h^{n-1}}{\Delta t}, v_h\Big) + a(u_h^n, u_h^n, v_h) = (f(u_h^n), v_h), \quad \forall\ v_h \in V_h, \tag{2.4}$$

where $u_h^n = u_h(t_n)$, and

$$\begin{aligned} a(u_h, u_h, v_h) &= \sum_{x_i \in \mathcal{N}_h} \int_{\partial V_i} (a(u_h)\nabla u_h)\cdot \mathbf{n} v_h ds \\ &= \sum_{x_i \in \mathcal{N}_h} v_h(x_i) \int_{\partial V_i} (a(u_h)\nabla u_h)\cdot \mathbf{n} ds. \end{aligned}$$

Define the discrete norm

$$|||u_h|||_0^2 = (u_h, I_h^* u_h), \quad \forall\ u_h \in U_h,$$

which is equivalent to the standard L^2-norm (see [23]), namely, there exist two positive constants C_*, C^* such that

$$C_*||u_h||_0 \le |||u_h|||_0 \le C^*||u_h||_0, \quad \forall\ u_h \in U_h. \tag{2.5}$$

To proceed the theoretical analysis for (2.4), the following discrete Gronwall lemma is needed.

Lemma 2.1 ([28]). *Let C_0 and a_k, b_k, c_k, d_k, for integers $k \ge 0$, be non-negative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \le \Delta t \sum_{k=0}^{n-1} d_k a_k + \Delta t \sum_{k=0}^{n-1} c_k + C_0, \qquad \forall\ n \ge 1.$$

Then

$$a_n + \Delta t \sum_{k=0}^{n} b_k \le \Big(\Delta t \sum_{k=0}^{n-1} c_k + C_0\Big) \exp\Big(\Delta t \sum_{k=0}^{n-1} d_k\Big), \qquad \forall\ n \ge 1.$$

The following lemmas will play key roles in the convergence of analysis.

Lemma 2.2 ([5]). *For all $u_h, v_h \in U_h$, there exists a positive constant C, such that*

$$(u_h, I_h^* v_h) = (v_h, I_h^* u_h), \quad (u_h, I_h^* v_h) \leq C||u_h||_0||v_h||_0.$$

Lemma 2.3 ([13,14]). *Supposing that the partition T_h is regular, T_h^* is the corresponding dual partition. For all $w_h, u_h, v_h \in U_h$, there exist two positive constants α and C such that*

$$\begin{aligned}&\alpha||u_h||_1^2 \leq a(w_h, u_h, I_h^* u_h), \quad a(w_h, u_h, I_h^* v_h) \leq C||u_h||_1||v_h||_1,\\ &|a(w_h, u_h, I_h^* v_h) - a(w_h, v_h, I_h^* u_h)| \leq Ch||u_h||_1||v_h||_1.\end{aligned}$$

Lemma 2.4 ([13]). *Introducing an elliptic operator $P_h : C(\Omega) \to U_h$, which defined by*

$$a(u, P_h u - u, v_h) = 0, \quad \forall v_h \in V_h, \ 0 < t \leq T.$$

Then there exists a positive constant C such that

$$\begin{aligned}&\|\nabla P_h u\|_\infty \leq C, \quad \|u - P_h u\|_1 \leq Ch\|u\|_2, \quad \|u - P_h u\|_0 \leq Ch^2\|u\|_3,\\ &\|(u - P_h u)_t\|_1 \leq Ch\Big(\|u\|_2 + \|u_t\|_2\Big), \qquad \|(u - P_h u)_t\|_0 \leq Ch^2(\|u\|_3 + \|u_t\|_3).\end{aligned}$$

Theorem 2.5 ([34]). *Under the assumptions of Lemma 2.3, if $u \in H^2(\Omega)$ and $w \in W^{1,\infty}(\Omega)$, then, there exists a positive constant C such that*

$$|a(u - u_h, w, I_h^* v_h)| \leq C\Big(h^2||u||_2 + ||u - u_h||_0\Big)||w||_{1,\infty}||v_h||_1, \quad \forall\ u_h, v_h \in U_h.$$

We end this section by introducing the following lemma, which can be found in [35].

Lemma 2.6. *It holds that*

$$(\overline{u}, \overline{u}) - (u, u) \leq C\Delta t(u, u), \qquad \forall\ u \in H_0^1(\Omega),$$

where $\overline{u} = u(x - u(x,t)\Delta t)$.

3. Error Estimates

This section is devoted to derive the error estimates of numerical solution in the characteristic finite volume scheme (2.4). As usual, we write the error $e = u^n - u_h^n$ as a sum of two terms

$$u^n - u_h^n = (u^n - P_h u^n) + (P_h u^n - u_h^n) = \eta^n + \xi^n,$$

where $u^n = u(t_n)$ and P_h is defined by Lemma 2.4. Firstly, we present the error estimate in L^2-norm for problem (2.4).

Theorem 3.1. *Under the conditions (C_1)-(C_4) and assume that $u_h^0 = P_h u_0$, the numerical solution u_h^n of problem (2.4) satisfies the following error estimate:*

$$\max_{1\leq n\leq N} ||u - u_h^n||_0 \leq C(h^2 + \Delta t). \tag{3.1}$$

Proof. Denoting $\partial_t \xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$, subtracting (2.4) from (2.3), choosing $v = \xi^n$ in (2.3) and taking $v_h = I_h^* \xi^n$ in (2.4), we have

$$\begin{aligned}&(\partial_t \xi^n, I_h^* \xi^n) + a(u_h^n, \xi^n, I_h^* \xi^n) \qquad (3.2)\\ =& -\Big(\psi(x,t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, I_h^* \xi^n\Big) - \Big(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^* \xi^n\Big) - \Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, I_h^* \xi^n\Big)\\ & - a(u^n - u_h^n, P_h u^n, I_h^* \xi^n) - \Big(\frac{\xi^{n-1} - \overline{\xi}^{n-1}}{\Delta t}, I_h^* \xi^n\Big) + (f(u^n) - f(u_h^n), I_h^* \xi^n).\end{aligned}$$

Using the definition of $|||\cdot|||_0$ and Lemma 2.2, we obtain that

$$
\begin{aligned}
&(\partial_t\xi^n, I_h^*\xi^n) = \frac{1}{\Delta t}\Big(\xi^n - \xi^{n-1}, I_h^*\xi^n\Big)\\
&= \frac{1}{2\Delta t}\left(\xi^n - \xi^{n-1}, I_h^*\Big[(\xi^n + \xi^{n-1}) + (\xi^n - \xi^{n-1})\Big]\right)\\
&\geq \frac{1}{2\Delta t}\Big[(\xi^n, I_h^*\xi^n) - (\xi^{n-1}, I_h^*\xi^{n-1})\Big] \geq \frac{1}{2\Delta t}(|||\xi^n|||_0^2 - |||\xi^{n-1}|||_0^2).
\end{aligned}
\tag{3.3}
$$

Combining (3.2) with (3.3), multiplying $2\Delta t$ and summing (3.2) for n from 1 to l $(1 \leq l \leq N)$, and using (2.5) and Lemma 2.3, we have

$$
\begin{aligned}
&||\xi^l||_0^2 + 2\alpha\sum_{n=1}^{l}||\xi^n||_1^2\Delta t\\
&\leq -2\sum_{n=1}^{l}(\psi(x,t^n)\frac{\partial u^n}{\partial\tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, I_h^*\xi^n)\Delta t - 2\sum_{n=1}^{l}\Big(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^*\xi^n\Big)\Delta t\\
&\quad -2\sum_{n=1}^{l}\Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, I_h^*\xi^n\Big)\Delta t - 2\sum_{n=1}^{l}a(u^n - u_h^n, P_h u^n, I_h^*\xi^n)\Delta t\\
&\quad -2\sum_{n=1}^{l}\Big(\frac{\xi^{n-1} - \overline{\xi}^{n-1}}{\Delta t}, I_h^*\xi^n\Big)\Delta t + 2\sum_{n=1}^{l}\Big(f(u^n) - f(u_h^n), I_h^*\xi^n\Big)\Delta t\\
&:= \sum_{i=1}^{6}E_i.
\end{aligned}
\tag{3.4}
$$

Now, we estimate the right-hand terms of (3.4) one by one. For E_1, with the results provided in [27], we have

$$
\begin{aligned}
|E_1| &\leq 2\sum_{n=1}^{l}\|\psi(x,t^n)\frac{\partial u^n}{\partial\tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}\|_0\|I_h^*\xi^n\|_0\Delta t\\
&\leq C_1\Delta t^2\int_0^T\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 ds + C_2\sum_{n=1}^{l}\|\xi^n\|_0^2\Delta t.
\end{aligned}
\tag{3.5}
$$

By a trick used in [8], Cauchy inequality and Lemma 2.4, we have

$$
\begin{aligned}
|E_2| &= \left|2\sum_{n=1}^{l}\Big(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^*\xi^n\Big)\Delta t\right| \leq 2\sum_{n=1}^{l}\|\xi^n\|_0\|\eta^n - \eta^{n-1}\|_0\\
&\leq 2\sum_{n=1}^{l}\|\xi^n\|_0\cdot\Big\|\int_{t_{n-1}}^{t_n}\frac{\partial\eta}{\partial t}\Big\|_0 ds = 2\sum_{n=1}^{l}\|\xi^n\|_0\cdot\Big\|\int_{t_{n-1}}^{t_n}(u - P_h u)_t\Big\|_0 ds\\
&\leq C_1\sum_{n=1}^{l}\|\xi^n\|_0\Delta t^{1/2}\Big(\int_{t_{n-1}}^{t_n}\|(u - P_h u)_t\|_0^2 ds\Big)^{\frac{1}{2}}\\
&\leq C_1\sum_{n=1}^{l}\|\xi^n\|_0\Delta t + C_2\int_0^T\|(u - P_h u)_t\|_0^2 ds\\
&\leq C_1\sum_{n=1}^{l}\|\xi^n\|_0\Delta t + C_2 h^4\int_0^T\Big(\|u_t\|_3^2 + \|u\|_3^2\Big)ds.
\end{aligned}
\tag{3.6}
$$

With a similar treatment for E_2, we can obtain

$$\begin{aligned}|E_3| &= \left|2\sum_{n=1}^{l}\Big(\frac{\eta^{n-1}-\overline{\eta}^{n-1}}{\Delta t}, I_h^*\xi^n\Big)\Delta t\right| \le C\sum_{n=1}^{l}\|\xi^n\|_1\|\eta^{n-1}-\overline{\eta}^{n-1}\|_{H^{-1}(\Omega)}\\ &\le C\sum_{n=1}^{l}\|\xi^n\|_1\|\eta^{n-1}\|_0\Delta t \le \frac{\alpha}{4}\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t + C_1\sum_{n=1}^{l}\|u^{n-1}-P_hu^{n-1}\|_0^2\Delta t\\ &\le \frac{\alpha}{4}\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t + C_1h^4\int_0^T\Big(\|u_t\|_3^2+\|u\|_3^2\Big)ds. \end{aligned} \tag{3.7}$$

Using the conditions of (C_1)-(C_2), the triangular inequality, Lemma 2.4 and Theorem 2.5, we have

$$\begin{aligned}|E_4| &= \left|2\sum_{n=1}^{l}a(u^n-u_h^n, P_hu^n, I_h^*\xi^n)\Delta t\right|\\ &\le \sum_{n=1}^{l}C(h^2||u^n||_2+||u^n-u_h^n||_0)||P_hu^n||_{1,\infty}||\xi^n||_1\Delta t\\ &\le C_1\sum_{n=1}^{l}h^2(\|u^n\|_3+\|u_t^n\|_3)||u^n||_{1,\infty}||\xi^n||_1\Delta t + C_2\sum_{n=1}^{l}||\xi^n||_0||u^n||_{1,\infty}||\xi^n||_1\Delta t\\ &\le C_1h^4\int_0^T(\|u_t\|_3^2+\|u\|_3^2)ds + \frac{\alpha}{4}\sum_{n=1}^{l}||\xi^n||_1^2\Delta t + C_2\sum_{n=1}^{l}||\xi^n||_0^2\Delta t. \end{aligned} \tag{3.8}$$

Similarly, we can obtain

$$\begin{aligned}|E_5| &= \left|2\sum_{n=1}^{l}\Big(\frac{\xi^{n-1}-\overline{\xi}^{n-1}}{\Delta t}, I_h^*\xi^n\Big)\Delta t\right| \le C\sum_{n=1}^{l}\|\xi^n\|_1\|\xi^{n-1}-\overline{\xi}^{n-1}\|_{H^{-1}(\Omega)}\\ &\le C\sum_{n=1}^{l}\|\xi^n\|_1\|\xi^{n-1}\|_0\Delta t \le \frac{\alpha}{4}\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t + C_1\sum_{n=1}^{l}\|\xi^{n-1}\|_0^2\Delta t. \end{aligned} \tag{3.9}$$

For E_6, Using the condition (C_1) and Lemma 2.4 yields

$$\begin{aligned}|E_6| &= \left|2\sum_{n=1}^{l}(f(u^n)-f(u_h^n), I_h^*\xi^n)\Delta t\right|\\ &\le C_1\sum_{n=1}^{l}\Big(||\xi^n||_0^2+||\eta^n||_0^2\Big)\Delta t + C_2\sum_{n=1}^{l}||\xi^n||_0^2\Delta t\\ &\le C_1h^4\int_0^T\Big(\|u_t\|_3^2+\|u\|_3^2\Big)ds + C_2\sum_{n=1}^{l}||\xi^n||_0^2\Delta t. \end{aligned} \tag{3.10}$$

Combining the estimates for E_1 to E_6 with (3.4), we arrive at

$$\begin{aligned}&||\xi^l||_0^2 + 2\alpha\sum_{n=1}^{l}||\xi^n||_1^2\Delta t\\ \le\ & C_1h^4\int_0^T\Big(\|u_t\|_3^2+\|u\|_3^2\Big)ds + C_2\Delta t^2\int_0^T\Big\|\frac{\partial^2 u}{\partial\tau^2}\Big\|_0^2 ds\\ &+C_3\sum_{n=1}^{l}||\xi^{n-1}||_0^2\Delta t + \alpha\sum_{n=1}^{l}||\xi^n||_1^2\Delta t. \end{aligned} \tag{3.11}$$

Putting the last term into the left side of (3.11) and applying Lemma 2.1 give

$$||\xi^l||_0^2 + \alpha \sum_{n=1}^{l} ||\xi^n||_1^2 \Delta t \leq C_1 h^4 \int_0^T \Big(\|u_t\|_3^2 + \|u\|_3^2\Big) ds + C_2 \Delta t^2 \int_0^T \Big\|\frac{\partial^2 u}{\partial \tau^2}\Big\|_0^2 ds.$$

This together with the triangular inequality and Lemma 2.4, the desired result (3.1) is obtained. □

Remark 3.1. From Theorem 3.1, we can see that the CFVM is of only first order in Δt. To balance the spatial and temporal errors, one should choose $\Delta t = \mathcal{O}(h^2)$, which is a restriction to the CFVM. Hence, in the proof of the following theorems, we will demonstrate $\Delta t = \mathcal{O}(h^2)$ is reasonable.

Remark 3.2. As noted in (3.11), the term $\|\frac{\partial^2 u}{\partial \tau^2}\|_0$ appears in the error estimates in Theorem 3.1, instead of the term $\|\frac{\partial^2 u}{\partial t^2}\|_0$. The former is much smaller than the later for an advection-dominated problem. Therefore, the boundness of $\|\frac{\partial^2 u}{\partial \tau^2}\|_0$ can be controlled by $\|\frac{\partial^2 u}{\partial t^2}\|_0$ under the some assumptions about the exact solution u.

Next, we present the H^1-norm error estimate for problem (1.1) in characteristic finite volume scheme (2.4).

Theorem 3.2. *Assume that the conditions of Theorem* 3.1 *are valid. If* $u_h^0 = P_h u_0$, *and* $\Delta t = \mathcal{O}(h^2)$, *then the solution* u_h^n *of problem* (2.4) *satisfies*

$$\max_{1\leq n\leq N} ||u - u_h^n||_1 \leq C(h + \Delta t). \tag{3.12}$$

Proof. We obtain the following error equation by choosing $v = \partial_t \xi^n$ in (2.3) and $v_h = I_h^* \partial_t \xi^n$ in (2.4), respectively

$$\begin{aligned}
&\Big(\frac{\xi^n - \xi^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\Big) + a(u_h^n, \xi^n, I_h^* \partial_t \xi^n)\\
=& -\Big(\psi(x, t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\Big) - \Big(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\Big)\\
&-\Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\Big) - a(u^n - u_h^n, P_h u^n, I_h^* \partial_t \xi^n)\\
&-\Big(\frac{\xi^{n-1} - \overline{\xi}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\Big) + \big(f(u^n) - f(u_h^n), I_h^* \partial_t \xi^n\big).
\end{aligned} \tag{3.13}$$

It follows from Lemma 2.3 and the inequality $a(a-b) \geq \frac{1}{2}(a^2 - b^2)$ that

$$\begin{aligned}
a(u_h^n, \xi^n, I_h^* \partial_t \xi^n) \geq& \frac{1}{2\Delta t}\Big[a(u_h^n, \xi^n, I_h^* \xi^n) - a(u_h^n, \xi^{n-1}, I_h^* \xi^{n-1})\Big]\\
&-\frac{1}{2}\Big[a(u_h^n, \partial_t \xi^n, I_h^* \xi^n) - a(u_h^n, \xi^n, I_h^* \partial_t \xi^n)\Big].
\end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), testing (3.13) against Δt and summing over n from 1 to l $(1 \leq$

$l \le N$), and using (2.5) and Lemma 2.3, we have

$$\begin{aligned}
&\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t+\frac{\alpha}{2}||\xi^l||_1^2\\
\le& -\sum_{n=1}^{l}\Big(\psi(x,t^n)\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t},I_h^*\partial_t\xi^n\Big)\Delta t-\sum_{n=1}^{l}\Big(\frac{\eta^n-\eta^{n-1}}{\Delta t},I_h^*\partial_t\xi^n\Big)\Delta t\\
&-\sum_{n=1}^{l}\Big(\frac{\eta^{n-1}-\overline{\eta}^{n-1}}{\Delta t},I_h^*\partial_t\xi^n\Big)\Delta t-\sum_{n=1}^{l}a(u^n-u_h^n,P_hu^n,I_h^*\partial_t\xi^n)\Delta t\\
&+\frac{1}{2}\sum_{n=1}^{l}\Big[a(u_h^n,\partial_t\xi^n,I_h^*\xi^n)-a(u_h^n,\xi^n,I_h^*\partial_t\xi^n)\Big]\Delta t\\
&-\sum_{n=1}^{l}\Big(\frac{\xi^{n-1}-\overline{\xi}^{n-1}}{\Delta t},I_h^*\partial_t\xi^n\Big)\Delta t+\sum_{n=1}^{l}\Big(f(u^n)-f(u_h^n),I_h^*\partial_t\xi^n\Big)\Delta t\\
:=&\sum_{i=1}^{7}F_i. \qquad (3.15)
\end{aligned}$$

Now, we are in the position to estimate F_1 to F_6. First,

$$\begin{aligned}
|F_1|&\le C\sum_{n=1}^{l}\Big\|\psi(x,t^n)\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t}\Big\|_0\|I_h^*\partial_t\xi^n\|_0\Delta t\\
&\le C\Delta t^2\int_0^T\Big\|\frac{\partial^2u}{\partial\tau^2}\Big\|_0^2ds+\frac{1}{8}\sum_{n=1}^{l}\|\partial_t\xi^n\|_0^2\Delta t. \qquad (3.16)
\end{aligned}$$

For F_2 and F_3, by Lemma 2.4 and the techniques used in [8], we have

$$\begin{aligned}
&|F_2|+|F_3|\\
=&\Big|\sum_{n=1}^{l}\Big(\frac{\eta^n-\eta^{n-1}}{\Delta t},I_h^*\partial_t\xi^n\Big)\Delta t\Big|+\Big|\sum_{n=1}^{l}\Big(\frac{\eta^{n-1}-\overline{\eta}^{n-1}}{\Delta t},I_h^*\partial_t\xi^n\Big)\Delta t\Big|\\
\le& C_1\sum_{n=1}^{l}\|\partial_t\xi^n\|_0\|\eta^n-\eta^{n-1}\|_0+C_2\sum_{n=1}^{l}\|\partial_t\xi^n\|_0\|\nabla\eta^{n-1}\|_0\Delta t\\
\le& C_1\sum_{n=1}^{l}\|\partial_t\xi^n\|_0\cdot\Big\|\int_{t_{n-1}}^{t_n}\frac{\partial\eta}{\partial t}\Big\|_0ds+C_2\sum_{n=1}^{l}\|\partial_t\xi^n\|_0\|\nabla(u^{n-1}-P_hu^{n-1})\|_0\Delta t\\
\le& C_1\sum_{n=1}^{l}\|\partial_t\xi^n\|_0\cdot\Big\|\int_{t_{n-1}}^{t_n}(u-P_hu)_t\Big\|_0ds+C_2\sum_{n=1}^{l}h\|\partial_t\xi^n\|_0\|u\|_2\Delta t\\
\le& C_1\sum_{n=1}^{l}\|\partial_t\xi^n\|_0\Delta t^{1/2}\Big(\int_{t_{n-1}}^{t_n}\|(u-P_hu)_t\|_0^2ds\Big)^{\frac{1}{2}}+C_2\sum_{n=1}^{l}h\|u\|_2\|\partial_t\xi^n\|_0\Delta t\\
\le&\frac{1}{4}\sum_{n=1}^{l}\|\partial_t\xi^n\|_0^2\Delta t+C_1h^4\int_0^T(\|u_t\|_3^2+\|u\|_3^2)ds+C_2(T)\|u\|_2^2h^2. \qquad (3.17)
\end{aligned}$$

It follows from Lemmas 2.3 and 2.4, Theorem 2.5, and Cauchy inequalities that

$$\begin{aligned}|F_4| &\leq C\sum_{n=1}^{l}(h^2||u^n||_2+||u^n-u_h^n||_0)||P_hu^n||_{1,\infty}||\partial_t\xi^n||_1\Delta t\\ &\leq C(h^2||u^n||_2+h^2+\Delta t)\sum_{n=1}^{l}||P_hu^n||_{1,\infty}||\partial_t\xi^n||_1\Delta t\\ &\leq C(h^2+\Delta t)\sum_{n=1}^{l}||P_hu^n||_{1,\infty}h^{-1}||\partial_t\xi^n||_0\Delta t\\ &\leq C(h+h^{-1}\Delta t)^2\sum_{n=1}^{l}||P_hu^n||_{1,\infty}^2\Delta t+\frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t. \end{aligned}\tag{3.18}$$

Similarly, we have

$$|F_5|\leq\frac{Ch}{2}\sum_{n=1}^{l}||\xi^n||_1||\partial_t\xi^n||_1\Delta t\leq C\sum_{n=1}^{l}||\xi^n||_1^2\Delta t+\frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t.\tag{3.19}$$

For F_6, applying Theorem 3.1 yields

$$\begin{aligned}|F_6| &\leq\sum_{n=1}^{l}\left\|\frac{\xi^{n-1}-\overline{\xi}^{n-1}}{\Delta t}\right\|_0\|I_h^*\partial_t\xi^n\|_0\Delta t\leq C\sum_{n=1}^{l}||\nabla\xi^{n-1}||_0||\partial_t\xi^n||_0\Delta t\\ &\leq C\sum_{n=1}^{l}||\xi^{n-1}||_1^2\Delta t+\frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t.\end{aligned}\tag{3.20}$$

For F_7, it follows from the Theorem 3.1, condition (C_1) and Young inequality that

$$\begin{aligned}|F_7| &\leq C\sum_{n=1}^{l}||u^n-u_h^n||_0^2\Delta t+\frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t\\ &\leq C(T)(h^4+\Delta t^2)+\frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t.\end{aligned}\tag{3.21}$$

Combining (3.16)-(3.21) with (3.15), one gets

$$\begin{aligned}&\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t+\frac{\alpha}{2}||\xi^l||_1^2\\ &\leq C_1h^4\int_0^T\Big(||u_t||_3^2+\|u\|_3^2\Big)dt+C_2h^2\int_0^T||u||_2^2dt+C_3\Delta t^2\int_0^T\|\partial_{\tau\tau}u\|_0^2dt\\ &\quad+C_4\sum_{n=1}^{l}||\xi^{n-1}||_1^2\Delta t+C_5\sum_{n=1}^{l}\Big(h^2+h^{-2}\Delta t^2\Big)\Delta t+\frac{1}{2}\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t.\end{aligned}\tag{3.22}$$

Under the restriction $\Delta t=\mathcal{O}(h^2)$, putting the last term into the left side of (3.22), and applying Lemma 2.1, we arrive at

$$\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\Delta t+||\xi^l||_1^2\leq C(h^2+\Delta t^2).$$

The desired estimate follows from the triangular inequality and Lemma 2.4. $\square$

4. Two-Grid Characteristic Finite Volume Approximations

From now on, H and $h \ll H$ will be two real positive parameters tending to 0. Recall a coarse mesh triangulation of $T_H(\Omega)$ of Ω is given in Section 2. A fine mesh triangulation $T_h(\Omega)$ is generated by a mesh refinement process to $T_H(\Omega)$. The space $U_H \subset U_h$ is based on the triangulations $T_H(\Omega)$ and $T_h(\Omega)$, respectively. With above spaces, we consider the following two-grid characteristic finite volume methods.

4.1. Oseen two-grid characteristic finite volume approximation

Algorithm 4.1.
Step I. Solve the nonlinear parabolic problem on a coarse mesh, i.e. find $u_H^{no} \in U_H$ ($n = 1, 2, \ldots$) on coarse grid T_H, such that for all $v_H \in V_H$

$$\begin{cases} (\frac{u_H^{no}-\overline{u}_H^{n-1o}}{\Delta t}, v_H) + a(u_H^{no}, u_H^{no}, v_H) = (f(u_H^{no}), v_H), \\ u_H^{0o} = P_H u_0. \end{cases} \tag{4.1}$$

Step II. Solve a linear problem on the fine grid T_h, $\forall\ v_h \in V_h$, find $u_h^{no} \in U_h$ ($n = 1, 2, \ldots$), such that

$$\begin{cases} (\frac{u_h^{no}-\overline{u}_h^{n-1o}}{\Delta t}, v_h) + a(u_H^{no}, u_h^{no}, v_h) = (f(u_H^{no}) + f'(u_H^{no})(u_h^{no} - u_H^{no}), v_h), \\ u_h^{0o} = P_h u_0. \end{cases} \tag{4.2}$$

Now, we consider the convergence of u_h^{no} to u. From (2.3) and (4.2), we obtain the following error equation for any $v_h \in V_h$

$$\begin{aligned} &\Big(\frac{\xi^{no} - \xi^{n-1o}}{\Delta t}, v_h\Big) + a(u_H^{no}, \xi^{no}, v_h) \\ =& -\Big(\psi(x, t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, v_h\Big) - \Big(\frac{\eta^n - \eta^{n-1}}{\Delta t}, v_h\Big) \\ &-\Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, v_h\Big) - a(u^n - u_H^{no}, P_h u^n, v_h) - \Big(\frac{\xi^{n-1o} - \overline{\xi}^{n-1o}}{\Delta t}, v_h\Big) \\ &+(f(u^n) - f(u_H^{no}) - f'(u_H^{no})(u_h^{no} - u_H^{no}), v_h). \end{aligned} \tag{4.3}$$

Theorem 4.1. *Assume that the conditions (C_1)-(C_4) are valid and u_h^{no} is the solution of the Oseen two-grid characteristic finite volume scheme (4.2). If $u_h^0 = P_h u_0$, and $\Delta t = \mathcal{O}(h^2)$, then*

$$\max_{1 \le n \le N} ||u - u_h^{no}||_1 \le C\Big(h + H^2 + \Delta t\Big). \tag{4.4}$$

Proof. Denoting $\partial_t \xi^{no} = \frac{\xi^{no} - \xi^{n-1o}}{\Delta t}$ and choosing $v_h = I_h^* \partial_t \xi^{no}$ in (4.3), we get

$$\begin{aligned} &(\partial_t \xi^{no}, I_h^* \partial_t \xi^{no}) + a(u_H^{no}, \xi^{no}, I_h^* \partial_t \xi^{no}) \\ =& -\Big(\psi(x, t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\Big) - a(u^n - u_H^{no}, P_h u^n, I_h^* \partial_t \xi^{no}) \\ &-(\partial_t \eta^n, I_h^* \partial_t \xi^{no}) - \Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^{no}\Big) - \Big(\frac{\xi^{n-1o} - \overline{\xi}^{n-1o}}{\Delta t}, I_h^* \partial_t \xi^{no}\Big) \\ &+(f(u^n) - f(u_H^{no}) - f'(u_H^{no})(u_h^{no} - u_H^{no}), I_h^* \partial_t \xi^{no}). \end{aligned} \tag{4.5}$$

Using the same techniques as used in obtaining (3.14) and (4.5) gives

$$
\begin{aligned}
&(\partial_t\xi^{no}, I_h^*\partial_t\xi^{no}) + \frac{1}{2\Delta t}\Big(a(u_H^{no}, \xi^{no}, I_h^*\xi^{no}) - a(u_H^{no}, \xi^{n-1o}, I_h^*\xi^{n-1o})\Big)\\
\leq& -\Big(\psi(x,t^n)\frac{\partial u^n}{\partial\tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, I_h^*\partial_t\xi^{no}\Big) - a(u^n - u_H^{no}, P_h u^n, I_h^*\partial_t\xi^{no})\\
&-(\partial_t\eta^n, I_h^*\partial_t\xi^{no}) - \Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, I_h^*\partial_t\xi^{no}\Big) - \Big(\frac{\xi^{n-1o} - \overline{\xi}^{n-1o}}{\Delta t}, I_h^*\partial_t\xi^{no}\Big)\\
&+\frac{1}{2}\Big(a(u_H^{no}, \partial_t\xi^{no}, I_h^*\xi^{no}) - a(u_H^{no}, \xi^{no}, I_h^*\partial_t\xi^{no})\Big)\\
&+\Big(f(u^n) - f(u_H^{no}) - f'(u_H^{no})(u_h^{no} - u_H^{no}), I_h^*\partial_t\xi^{no}\Big)\\
:=& \sum_{i=1}^{7} G_i.
\end{aligned}\tag{4.6}
$$

Multiply (4.6) by Δt and summing it for n from 1 to l, $1 \leq l \leq N$. For G_1 and G_3-G_6, under the condition (C_2), by Lemmas 2.3 and 2.4, Theorem 2.5 and Cauchy inequality, we can estimate them as in Theorem 3.2. For G_2, under the restriction $\Delta t = \mathcal{O}(h^2)$, we can have

$$
\begin{aligned}
|G_2| \leq& \sum_{n=1}^{l} C\Big(H^2||u^n||_2 + ||u^n - u_H^{no}||_0\Big)||P_h u^n||_{1,\infty}||\partial_t\xi^{no}||_1\Delta t\\
\leq& \sum_{n=1}^{l} C\Big(H^2||u^n||_2 + ||u^n - u_H^{no}||_0\Big)||P_h u^n||_{1,\infty}||\partial_t\xi^{no}||_0 h^{-1}\Delta t\\
\leq& C(H^2 + \Delta t)\sum_{n=1}^{l}||\nabla P_h u^n||_{0,\infty}||\partial_t\xi^{no}||_0 h^{-1}\Delta t\\
\leq& C(H^2 + \Delta t)^2 + \frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^{no}||_0^2 h^{-2}\Delta t^2\\
\leq& C(H^4 + \Delta t^2) + \frac{1}{8}\sum_{n=1}^{l}||\partial_t\xi^{no}||_0^2\Delta t.
\end{aligned}\tag{4.7}
$$

For G_7, using the condition (C_3), the proof provided in [4] and Lemma 2.3 gives

$$
\begin{aligned}
&\Big(f(u^n) - f(u_H^{no}) - f'(u_H^{no})(u_h^{no} - u_H^{no}), I_h^*\partial_t\xi^{no}\Big)\Delta t\\
\leq& C\Big(||\xi^{no}||_0^2 + ||\eta^n||_0^2\Big)\Delta t + C(H^2 + \Delta t)^2\Delta t + \frac{1}{8}||\partial_t\xi^{no}||_0^2\Delta t.
\end{aligned}\tag{4.8}
$$

It follows from the above estimates and inequality (4.6) that

$$
\begin{aligned}
&\sum_{n=1}^{l}||\partial_t\xi^{no}||_0^2\Delta t + \alpha||\xi^{lo}||_1^2\\
\leq& C_1 h^2\int_0^T\Big(||u_t||_3^2 + ||u||_3^2 + ||u^n||_2^2\Big)dt + C_2\Delta t^2\int_0^{t^l}\Big\|\frac{\partial^2 u}{\partial\tau^2}\Big\|_0^2 dt\\
&+C_3\sum_{n=1}^{l}||\xi^{n-1o}||_1^2\Delta t + \frac{1}{2}\sum_{n=1}^{l}||\partial_t\xi^{no}||_0^2\Delta t + C_4\sum_{n=1}^{l}\Big(H^4 + \Delta t^2\Big)\Delta t.
\end{aligned}\tag{4.9}
$$

By Lemma 2.1, we have

$$||\xi^{lo}||_1 \le C(h + H^2 + \Delta t), \tag{4.10}$$

where C is a constant dependent on $||u||_{L^2(0,T;H^3(\Omega))}, ||u_t||_{L^2(0,T;H^3(\Omega))}$, but independent of h and Δt.The proof is complete by combining the triangular inequality with Lemma 2.4. □

4.2. Newton two-grid characteristic finite volume approximation

Algorithm 4.2.
Step I: Solve the nonlinear parabolic problem on a coarse mesh, i.e., fine $u_H^{nn} \in U_H$ by (4.1).
Step II: Solve the general linear parabolic problem on a fine mesh, i.e., apply one Newton step to find $u_h^{nn} \in U_h$ such that for all $v_h \in V_h$

$$\begin{cases} (\frac{u_h^{nn}-\overline{u}_h^{n-1n}}{\Delta t}, v_h) + a(u_H^{nn}, u_h^{nn}, v_h) + a(u_h^{nn}, u_H^{nn}, v_h) \\ \qquad\qquad = (f(u_H^{nn}) + f'(u_H^{nn})(u_h^{nn} - u_H^{nn}), v_h) + a(u_H^{nn}, u_H^{nn}, v_h), \\ u_h^{0n} = P_h u_0, \end{cases} \tag{4.11}$$

Now, we consider the convergence of the Newton two-grid characteristic finite volume scheme (4.11). To do this, setting $\xi^{nn} = P_h u^n - u_h^{nn}$. Then, from (2.3) and (4.11), we obtain the following error equation for any $v_h \in V_h$

$$\begin{aligned} &\Big(\frac{\xi^{nn} - \overline{\xi}^{n-1n}}{\Delta t}, v_h\Big) + a(u_h^{nn}, \xi^{nn}, v_h) \\ =& -\Big(\psi(x,t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, v_h\Big) - \Big(\frac{\eta^n - \eta^{n-1}}{\Delta t}, v_h\Big) - \Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, v_h\Big) \\ &-\Big(\frac{\xi^{n-1n} - \overline{\xi}^{n-1n}}{\Delta t}, v_h\Big) - a(u_h^{nn} - u_H^{nn}, u_h^{nn} - u_H^{nn}, v_h) - a(u^n - u_h^{nn}, u^n, v_h) \\ &+(f(u^n) - f(u_H^{nn}) - f'(u_H^{nn})(u_h^{nn} - u_H^{nn}), v_h). \end{aligned} \tag{4.12}$$

Theorem 4.2. *Assume that the conditions* (C_1)*-*(C_4) *are valid and* u_h^{nn} *be the solution of Newton two-grid characteristic finite volume algorithm* (4.11)*. If* $u_h^0 = P_h u_0$*, and* $\Delta t = \mathcal{O}(h^2)$*, then*

$$\max_{1\le n\le N} ||u - u_h^{nn}||_1 \le C\Big(h + |\log h|^{1/2} H^3 + \Delta t\Big).$$

Proof. Choosing $v_h = I_h^* \partial_t \xi^{nn}$ in (4.12) and using the trick as adopted in Theorem 3.2, we have

$$\begin{aligned} &(\partial_t \xi^{nn}, I_h^* \partial_t \xi^{nn}) + \frac{1}{2\Delta t}\Big(a(u_H^{nn}, \xi^{nn}, I_h^* \xi^{nn}) - a(u_H^{nn}, \xi^{n-1n}, I_h^* \xi^{n-1n})\Big) \\ \le& -\Big(\psi(x,t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^{nn}\Big) - \Big(\frac{\eta^{n-1} - \overline{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^{nn}\Big) - \Big(\frac{\xi^{n-1n} - \overline{\xi}^{n-1n}}{\Delta t}, I_h^* \partial_t \xi^{nn}\Big) \\ &-(\partial_t \eta^n, I_h^* \partial_t \xi^{nn}) - a(u^n - u_h^{nn}, P_h u^n, I_h^* \partial_t \xi^{nn}) - a(u_h^{nn} - u_H^{nn}, u_h^{nn} - u_H^{nn}, I_h^* \partial_t \xi^{nn}) \\ &+\frac{1}{2}\Big[a(u_H^{nn}, \partial_t \xi^{nn}, I_h^* \xi^{nn}) - a(u_H^{nn}, \xi^{nn}, I_h^* \partial_t \xi^{nn})\Big] \\ &+(f(u^n) - f(u_H^{nn}) - f'(u_H^{nn})(u_h^{nn} - u_H^{nn}), I_h^* \partial_t \xi^{nn}) \\ =& \sum_{i=1}^{8} D_i. \end{aligned} \tag{4.13}$$

Multiplying (4.13) by Δt and summing it for n from 1 to l, $(1 \le l \le N)$. For D_1-D_4 and D_7-D_8, under the condition of (C_2), by Lemmas 2.3 and 2.4 and Cauchy inequality, we can estimate them, using the same techniques as used in the proof of Theorem 3.2. For D_5, we have

$$\begin{aligned}|D_5| &\le \sum_{n=1}^{l} C\Big(h^2||u^n||_2 + ||u^n - u_h^{nn}||_0\Big)||P_h u^n||_{1,\infty}||\partial_t \xi^{nn}||_1 \Delta t \\ &\le \sum_{n=1}^{l} C\Big(h^2||u^n||_2 + ||u^n - u_h^{nn}||_0\Big)||\nabla P_h u^n||_{0,\infty}||\partial_t \xi^{nn}||_0 h^{-1} \Delta t \\ &\le C(h^4 + \Delta t^2) + \frac{1}{8}\sum_{n=1}^{l}||\partial_t \xi^{nn}||_0^2 \Delta t. \end{aligned} \tag{4.14}$$

For D_6, it follows from the triangular inequality and Theorem 2.5 that

$$\begin{aligned}&|a(u_h^{nn} - u_H^{nn}, u_h^{nn} - u_H^{nn}, I_h^* \partial_t \xi^{nn})| \\ \le\ & |a(u_h^{nn} - u^n, u_h^{nn} - u_H^{nn}, I_h^* \partial_t \xi^{nn})| + |a(u^n - u_H^{nn}, u_h^{nn} - u_H^{nn}, I_h^* \partial_t \xi^{nn})| \\ \le\ & \sum_{n=1}^{l} C_1\Big(h^2||u^n||_2 + ||u^n - u_h^{nn}||_0\Big)||u_h^{nn} - u_H^{nn}||_{1,\infty}||\partial_t \xi^{nn}||_1 \Delta t \\ &\quad + \sum_{n=1}^{l} C_2\Big(H^2||u^n||_2 + ||u^n - u_H^{nn}||_0\Big)||u_h^{nn} - u_H^{nn}||_{1,\infty}||\partial_t \xi^{nn}||_1 \Delta t \\ \le\ & \sum_{n=1}^{l} C_1\Big(h^2||u^n||_2 + ||u^n - u_h^{nn}||_0\Big)|\log h|^{1/2}||u_h^{nn} - u_H^{nn}||_1||\partial_t \xi^{nn}||_0 h^{-1} \Delta t \\ &\quad + \sum_{n=1}^{l} C_2\Big(H^2||u^n||_2 + ||u^n - u_H^{nn}||_0\Big)|\log h|^{1/2}||u_h^{n} - u_H^{nn}||_1||\partial_t \xi^{nn}||_0 h^{-1} \Delta t \\ \le\ & \sum_{n=1}^{l} C_1\Big(h^2 + \Delta t\Big)|\log h|^{1/2}\Big(h + H + \Delta t\Big)||\partial_t \xi^{nn}||_0 h^{-1} \Delta t \\ &\quad + \sum_{n=1}^{l} C_2\Big(H^2 + \Delta t\Big)|\log h|^{1/2}\Big(h + H + \Delta t\Big)||\partial_t \xi^{nn}||_0 h^{-1} \Delta t \\ \le\ & C\Big(h^4 H^2 + H^6\Big)|\log h| + \frac{1}{8}\sum_{n=1}^{l}||\partial_t \xi^{nn}||_0^2 \Delta t. \end{aligned} \tag{4.15}$$

Combining (4.13) with above estimates and applying Lemma 2.1 and triangular inequality, we obtain the desired results. □

5. Numerical Experiments

In order to gain insights on the established theoretical results in Sections 3 and 4, we present some numerical experiments in this section. Our main interest is to verify the performances of the Oseen and Newton two-grid characteristic finite volume algorithm (4.2) and (4.11). In all experiments, $\Omega = [0,1] \times [0,1]$, $T = 0.1$, $\Delta t = h^2$. The mesh consists of triangular elements. In order to show the prominent features of the two-grid characteristic finite volume method, we compare our schemes (4.2) and (4.11) with the CFVM (2.4) for the nonlinear problem (1.1). In each time iterative interval $[t_{m-1}, t_m]$, the stopping criterion

$$\left(\sum_{i=1}^{(N+1)^2} (u_{h,i}^m - u_{h,i}^{m-1})^2 \right)^{\frac{1}{2}} \leq 10^{-4}$$

is employed, where N is the number of nodes in each orientation, m is the step of the iterative and initial value $u_h^0 = u_h(0)$. u_h^{no} and u_h^{nn} be the numerical solution which obtained by using the Oseen and Newton two-grid CFVM methods, respectively. The experimental rates of convergence with respect to the mesh size h are calculated by the formula $\frac{\log(E_i/E_{i+1})}{\log(h_i/h_{i+1})}$, where E_i and E_{i+1} are the relative errors corresponding to the mesh of sizes h_i and h_{i+1}, respectively.

We choose the coefficients of problem (1.1) are $a(u) = u$, $b_1(u) = b_2(u) = u$. The initial-boundary values and $f(u)$ are determined by the exact solution $u = e^{-t}x(1-x)y(1-y)$. The CPU time and relative errors are listed in Tables 5.1-5.3 for the CFVM and two-level CFVM methods with some h and H. It is need to explain that the coarse mesh sizes in Newton two-grid CFVM method, from Theorem 4.2, we should choose $H = \sqrt[3]{h|\log h|^{-\frac{1}{2}}}$, in order to put the suitable nodes N, we take the numbers of mesh points as a integer of $N = [\frac{1}{H}]$.

Table 5.1: Characteristic finite volume method for nonlinear parabolic equations.

$\frac{1}{h}$	$\frac{\|u-u_h^n\|_0}{\|u\|_0}$	$\frac{\|u-u_h^n\|_1}{\|u\|_1}$	u_{L^2} rate	u_{H^1} rate	CPU(s)
9	0.0331786	0.413652	/	/	0.857
16	0.0112041	0.238276	1.8868	0.9587	7.469
25	0.0049335	0.157435	1.8379	0.9286	43.984
36	0.0024609	0.110440	1.9074	0.9723	195.329
49	0.0013811	0.0813872	1.8736	0.9901	699.141

Table 5.2: Oseen two-level characteristic finite volume method for nonlinear parabolic equations.

$\frac{1}{h}$	$\frac{1}{H}$	$\frac{\|u-u_h^{no}\|_0}{\|u\|_0}$	$\frac{\|u-u_h^{no}\|_1}{\|u\|_1}$	u_{L^2} rate	u_{H^1} rate	CPU(s)
9	3	0.0321762	0.429481	/	/	0.469
16	4	0.0116502	0.252669	1.7657	0.9220	3.656
25	5	0.0051738	0.170766	1.8188	0.8779	20.510
36	6	0.0026148	0.123599	1.8714	0.8865	88.313
49	7	0.0014607	0.092588	1.8886	0.9370	295.438

Table 5.3: Newton two-level characteristic finite volume method for nonlinear parabolic equations.

$\frac{1}{h}$	$\frac{1}{H}$	$\frac{\|u-u_h^{nn}\|_0}{\|u\|_0}$	$\frac{\|u-u_h^{nn}\|_1}{\|u\|_1}$	u_{L^2} rate	u_{H^1} rate	CPU(s)
9	2	0.0233606	0.385462	/	/	0.406
16	3	0.0084201	0.236872	1.7735	0.8463	3.375
25	4	0.0037730	0.162319	1.7988	0.8469	19.438
36	5	0.0019677	0.119149	1.7853	0.8479	84.079
49	6	0.0011379	0.090306	1.7765	0.8990	292.859

From Tables 5.1-5.3, we can see that the numerical results coincide with the theoretical analysis, and both the Oseen and Newton two-level characteristic finite volume methods spend less time than CFVM, that is to say, our algorithms are effective for saving a large amount of computational time and still keeping good precise.

6. Conclusions

In this paper, we consider the two-grid characteristic finite volume methods for the nonlinear parabolic problem. The L^2 and H^1-norm error estimates for the modified method of characteristic finite volume are derived under some assumptions. Furthermore, for two-grid algorithms, by using Taylor expression and the known solution u_H^n, which obtained in coarse mesh, the nonlinear system transforms into a linear system, which is much easier to be solved than the origin problem, some numerical results are provided to confirm the effectiveness of our methods.

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References

[1] R. Bank and D. Rose, Some error estimates for the box method, *SIAM J. Numer. Anal.*, **24**:4 (1987), 777-787.

[2] C.J. Bi and V.Ginting, Two-grid finite volume element method for linear and nonlinear elliptic problems, *Numer. Math.*, **108**:2 (2007), 177-198.

[3] Z.Q. Cai, On the finite volume method, *Numer. Math.*, **58**:7 (1991), 713-735.

[4] C.J. Chen, M. Yang and C.J. Bi, Two-grid methods for finite volume element approximations of nonlinear parabolic equations, *J. Comput. Appl. Math.*, **228**:1 (2009), 123-132.

[5] S. Chou and Q. Li, Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problem: A unified approach, *Math. Comp.*, **69**:229 (2000), 103-120.

[6] C. Dawson and M. Wheeler, Two-grid methods for mixed finite element approximations of nonlinear parabolic equations, *Contemp. Math.*, **180** (1994), 191-203.

[7] C. Dawson, M. Wheeler and C. Woodward, A two-grid finite difference scheme for nonlinear parabolic equations, *SIAM J. Numer. Anal.*, **35**:2 (1998), 435-452.

[8] J. Douglas and T. Russell, Numerical methods for convection-dominated diffusion problemsbased on combining the method of characteristics with finite element or finite difference procedures, *SIAM J. Numer. Anal.*, **19** (1982), 871-885.

[9] V. Dolejsi, M. Feistauer and C. Schwab, A finite volume discontinuous Galerkin scheme for nonlinear convection-diffusion problems, *CALCOLO*, **39**:1 (2002), 1-40.

[10] R. Ewing, O. Iliev and R. Lazarov, A modified finite volume approximation of second-order elliptic equations with discontinuous coefficients , *J. Sci. Comput.*, **23**:4 (2001), 1334-1350.

[11] R. Ewing, T. Lin and Y.P. Lin, On the accuracy of the finite volume element based on piecewise linear polynomials, *SIAM J. Numer. Anal.*, **39**:6 (2002), 1865-1888.

[12] R. Eymard, T. Gallouet and R. Herbin, Finite volume methods, Handbook of Numer.Anal., P.G. Ciarlet, J.L. Lions eds, (1997) 713-1020.

[13] F.Z. Gao, Finite volume element predictor-corrector method for a class of nonlinear parabolic systems, *Northeast Math. J.*, **21**:3 (2005), 305-314.

[14] F.Z. Gao and Y.R. Yuan, The characteristic finite volume element method for the nonlinear convection-dominated diffusion problem, *Comput. Math. Appl.*, **56**:1 (2008), 71-81.

[15] Y.N. He, Two-level method based on finite element and Crank-Nicolson extrapolation for the time-dependent Navier-Stokes equations, *SIAM J. Numer. Anal.*, **41**:4 (2003), 1263-1285.

[16] Y.N. He, A two-level finite element method for the nonstationary Navier-Stokes equations I: spatial discretization, *J. Comp. Math.*, **22**:1 (2004), 21-32.

[17] Y.N. He, H.L. Miao and C.F. Ren, A two-level finite element method for the nonstationary Navier-Stokes equations II: time discretization, *J. Comp. Math.*, **22**:1 (2004), 33-54.

[18] Y.N. He and K. Liu, A Multi-level Finite element method in space-time for the Navier-Stokes equations, *Numer. Meth. Part. Differ. Equ.*, **21**:6 (2005), 1052-1078.

[19] J. Jones, Z.Q. Cai, S.McCormick and T.Russell, Control-volume mixed finite element methods, *Comput. Geosci.*, **1**:3 (1997), 289-315.

[20] Z. Kriva and K. Mikula, An adaptive finite volume scheme for solving nonlinear diffusion equations in image processing, *J. Vis. Commun. Image Represent.*, **13**:1 (2002), 22-35.

[21] R.H. Li, Generalized difference methods for a nonlinear Dirichlet problem, *SIAM J. Numer. Anal.*, **24**:1 (1987), 77-88.

[22] W. Layton and W. Lenferink, Two-level Picard and modified Picard methods for the Navier-Stokes equations, *Appl. Math. Comp.*, **69**:1 (1995), 263-274.

[23] R.H. Li, Z.Y. Chen and W. Wu, Generalized Difference Methods for Differential Equations, Marcel Dekker Inc., New York, 2000.

[24] M. Marion and J.C. Xu, Error estimates on a new nonlinear Galerkin method based on two-grid finite elements, *SIAM J. Numer. Anal.*, **32** (1995), 1170-1184.

[25] K. Morton, Numerical solution of convection-diffusion problems, London: Chapman & Hall, 1996.

[26] H. Rachford, Two-level discrete-time Galerkin approximations for second order nonlinear parabolic partial differential equations, *SIAM J. Numer. Anal.*, **10**:6 (1973), 1010-1026.

[27] T. Russell, Time stepping along characteristics with incomplete iteration for a Galerkin approximation of miscible displacement in porous media, *SIAM J. Numer. Anal.*, **22**:5 (1985), 970-1013.

[28] J. Shen, Long time stability ad convergence for fully dscretee nonlinear Galerkin methods, *Applicable Analysis*, **38** (1990), 201-229.

[29] E. Süli, convergence ad nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations, *Numer. Math.*, **53** (1988), 459-483.

[30] V. Thomée, Galerkin Finite Element Method for Parabolic Problems, Springer-Verlag, Berlin, 1997.

[31] H.J. Wu and R.H. Li, Error Estimates for fnite volume element methods for general second order elliptic problem. *Numer. Meth. Part. Differ. Equ.*, 19 (2003) 693-708.

[32] J.C. Xu, A novel two-grid method for semi-linear elliptic equations, *SIAM J. Sci. Comput.*, **15**:1 (1994), 231-237.

[33] J.C. Xu, Two-grid discretization techniques for linear and nonlinear PDEs, *SIAM J. Numer. Anal.*, **33**:5 (1996), 1759-1777.

[34] T. Zhang, H. Zhong and J. Zhao, A full discrete two-grid finite-volume method for a nonlinear parabolic problem, *Int. J. Comput. Math.*, **88** (2011), 1644-1663.

[35] T. Zhang, Z.Y. Si and Y.N. He, A stabilised characteristic finite element method for transient Navier-Stokes equations, *Int. J. Comput. Fluid Dyn.*, **24** (2010), 369-381.