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A New Homotopy Method for Nonlinear Complementarity Problems

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Abstract

In this paper, we present a new homotopy method for the nonlinear complementarity problems. Without the regularity or non-singulary assumptions for $\nabla F(x)$, we prove that our homotopy equations have a bounded solution curve. The numerical tests confirm the efficiency of our proposed method.

Keywords: Nonlinear complementarity problem (NCP); homotopy equations; bounded.
Mathematics subject classification: 90C33

1. Introduction

We are interested in finding a solution of the following nonlinear complementarity problem (NCP):

$$\begin{cases} \text{find} \quad x \in \mathbb{R}^n \\ \text{s. t.} \quad x_i \geq 0,\ F_i(x) \geq 0,\ x_i F_i(x) = 0,\ \text{for } i = 1, \cdots, n, \end{cases}$$

where $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuously differentiable.

There have been extensive studies for complementarity problems, see for example [5] and a recent review paper [7]. The NCP was introduced by Cottle [4] in his Ph.D. thesis in 1964. At the beginning, it was recognized that the NCP is a special case of a variation inequality problem [11]. The nonlinear complementarity problem and the linear complementarity problem, where $F(x)$ is an affine map, have been studied extensively, see, e.g., [2, 8, 10].

There exist many methods for solving the NCP, such as semi-smooth equation method [15], smoothing or non-smoothing Newton method [17, 19], neural network method [13, 14] etc. Homotopy method, as a kind of fixed-point method, had been developed in 1970s: Scarf [18] introduced the notion of primitive sets and described the first algorithm to approximate a fixed point of a continuous mapping; some powerful theoretical tools were available [3]; Eaves [6] introduced piecewise-linear maps into the computational fixed-point literature. The classical reference by Todd [20] and the text by Garcia and Zangwill

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[9] provided discussions of these techniques in detail. As a historical note, homotopy-type methods were directly applied to the linear complementarity problems with some success in [21,22], but [23] was the only one with homotopy method to solve NCP. In this paper, we can avoid the requirement that Jacobian matrix of $F(x)$ is regular as compared to [23].

The organization of the paper is as follows. In Section 2, we prove the existence and boundedness for the solution curve of the homotopy equation. We also give the conditions which ensure the end point of the solution curve to be the solution of NCP. In Section 3, some numerical examples are provided. In Section 4, some concluding remarks will be made.

2. Main results of a homotopy method for NCP

Suppose

$$K^{(M)} = \{x \in \mathbb{R}^n; 0 \leq x_i \leq M, i = 1, \cdots, n\},$$
$$\tilde{K}^{(M)} = \{x \in \mathbb{R}^n; 0 < x_i < M, i = 1, \cdots, n\},$$

and let $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x_i > 0, i = 1, \cdots, n\}$, $\mathbb{R}^n_{+} = \{x \in \mathbb{R}^n | x_i \geq 0, i = 1, \cdots, n\}$.

Lemma 2.1. (Parametric Form of Sard Theorem [1]) *Let $V \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$ be open sets, and let $F : V \times U \to \mathbb{R}^k$ be a C^r mapping, where $r > \max\{0, m-k\}$. If $0 \in \mathbb{R}^k$ is a regular value of F, then for almost all $a \in V$, 0 is a regular value of $F_a = F(a, \cdot)$.*

Lemma 2.2. (Inverse Image Theorem [16]) *If 0 is a regular value of the mapping F_a, then $F_a^{-1}(0)$ consists of some smooth manifolds.*

Lemma 2.3. (Classification Theorem of One-Dimensional Smooth Manifolds [16]) *A one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

In this paper, we consider the following homotopy equation:

$$H(w^{(0)}, w, t) = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} (1-t)(F(x) - y + z) + t(x - x^{(0)}) \\ -\begin{pmatrix} x_1 y_1 \\ \vdots \\ x_n y_n \end{pmatrix} + t \begin{pmatrix} x_1^{(0)} y_1^{(0)} \\ \vdots \\ x_n^{(0)} y_n^{(0)} \end{pmatrix} \\ -\begin{pmatrix} (M - x_1) z_1 \\ \vdots \\ (M - x_n) z_n \end{pmatrix} + t \begin{pmatrix} (M - x_1^{(0)}) z_1^{(0)} \\ \vdots \\ (M - x_n^{(0)}) z_n^{(0)} \end{pmatrix} \end{bmatrix} = 0, \quad (2.1)$$

where $x, y, z \in \mathbb{R}^n$, $w = (x, y, z)$, $w^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)})$ and $t \in (0, 1]$.

Proposition 2.1. *For every $M > 0$ and almost all $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, 0 is a regular value of the mapping $H_{w^{(0)}} = H(w^{(0)}, w, t) : \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0, 1] \to \mathbb{R}^{2n}$ and $H^{-1}_{w^{(0)}}(0) = \{(w, t) \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0, 1];\ H_{w(0)}(w, t) = 0\}$ is composed by some smooth and simple curves, with one of the curves has the initial point $(w^{(0)}, 1)$.*