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Generalized Jacobi and Gauss-Seidel Methods for Solving Linear System of Equations

Davod Khojasteh Salkuyeh*
(*Department of Mathematics, Mohaghegh Ardabili University, P.O. Box 56199-11367, Ardabil, Iran*
E-mail: khojaste@uma.ac.ir)

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Abstract

The Jacobi and Gauss-Seidel algorithms are among the stationary iterative methods for solving linear system of equations. They are now mostly used as preconditioners for the popular iterative solvers. In this paper a generalization of these methods are proposed and their convergence properties are studied. Some numerical experiments are given to show the efficiency of the new methods.

Keywords: Jacobi; Gauss-Seidel; generalized; convergence.
Mathematics subject classification: 65F10, 65F50

1. Introduction

Consider the linear system of equations

$$Ax = b, \tag{1.1}$$

where the matrix $A \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n$. Let A be a nonsingular matrix with nonzero diagonal entries and

$$A = D - E - F,$$

where D is the diagonal of A, $-E$ its strict lower part, and $-F$ its strict upper part. Then the Jacobi and the Gauss-Seidel methods for solving Eq. (1.1) are defined as

$$x_{k+1} = D^{-1}(E + F)x_k + D^{-1}b,$$
$$x_{k+1} = (D - E)^{-1}Fx_k + (D - E)^{-1}b,$$

respectively. There are many iterative methods such as GMRES [7] and Bi-CGSTAB [9] algorithms for solving Eq. (1.1) which are more efficient than the Jacobi and Gauss-Seidel methods. However, when these methods are combined with the more efficient methods, for example as a preconditioner, can be quite successful. For example see [4, 6]. It has been proved that if A is a strictly diagonally dominant (SDD) or irreducibly diagonally dominant, then the associated Jacobi and Gauss-Seidel iterations converge for any initial guess x_0 [6]. If A is a symmetric positive definite (SPD) matrix, then the Gauss-Seidel

*Corresponding author.

method also converges for any x_0 [1]. In this paper we generalize these two methods and study their convergence properties.

This paper is organized as follows. In Section 2, we introduce the new algorithms and verify their properties. Section 3 is devoted to the numerical experiments. In Section 4 some concluding remarks are also given.

2. Generalized Jacobi and Gauss-Seidel methods

Let $A = (a_{ij})$ be an $n \times n$ matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth $2m+1$ defined as

$$t_{ij} = \begin{cases} a_{ij}, & |\, i-j \,| \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the decomposition $A = T_m - E_m - F_m$ where $-E_m$ and $-F_m$ are the strict lower and upper part of the matrix $A_m - T_m$, respectively. In other words matrices T_m, E_m and F_m are defined as following

$$T_m = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1,1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix},$$

$$E_m = \begin{pmatrix} & & & \\ & & & \\ -a_{m+2,1} & & & \\ \vdots & \ddots & & \\ -a_{n,1} & \cdots & -a_{n-m-1,n} & \end{pmatrix},$$

$$F_m = \begin{pmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m-1,n} \\ & & & \\ & & & \end{pmatrix}.$$

Then we define the generalized Jacobi (GJ) and generalized Gauss-Seidel (GGS) iterative methods as follows

$$x_{k+1} = T_m^{-1}(E_m + F_m)x_k + T_m^{-1}b, \tag{2.1}$$

$$x_{k+1} = (T_m - E_m)^{-1}F_m x_k + (T_m - E_m)^{-1}b, \tag{2.2}$$