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The Fundamental Solutions of the Space, Space-Time Riesz Fractional Partial Differential Equations with Periodic Conditions

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Abstract

In this paper, the space-time Riesz fractional partial differential equations with periodic conditions are considered. The equations are obtained from the integral partial differential equation by replacing the time derivative with a Caputo fractional derivative and the space derivative with Riesz potential. The fundamental solutions of the space Riesz fractional partial differential equation (SRFPDE) and the space-time Riesz fractional partial differential equation (STRFPDE) are discussed, respectively. Using methods of Fourier series expansion and Laplace transform, we derive the explicit expressions of the fundamental solutions for the SRFPDE and the STRFPDE, respectively.

Keywords: Fundamental solution; Riesz potential; Caputo derivative; Fourier series; Laplace transform.
Mathematics subject classification: 35C05, 35K30

1. Introduction

Fractional-order partial differential equations are generalizations of classical partial differential equations. Recently, a growing number of works from various fields of science and engineering, such as physics, finance, hydrology, thermodynamics, etc., deal with dynamical systems described by fractional-order equations [1–7]. This is because fractional-order derivatives and integrals provide a powerful instrument for the description of memory and hereditary properties of different substances. Therefore, many authors want to find the fundamental solutions of some fractional-order differential equations by different ways. A Cauchy problem for the Riemann-Liouville fractional differential operator was studied by Luchko et al. [8], who got the solution of the problem using a Mikusiński-type operational calculus. Mainardi [9] considered the fundamental solutions for the time fractional diffusion-wave equation using Laplace transform. Furthermore, Mainardi et al. [10] discussed the fundamental solution of the space-time fractional diffusion equation using the

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Fourier and Laplace integral transforms and Mittag-Leffler functions, in which the fundamental solution can only be expressed as convolution form of the Green function and the initial value function, and then computed difficultly. Elizarraraz et al. [11] studied the solution of homogeneous differential equations associated with a fractional differential operator related to Weyl's operator in a vector space of generalized exponential polynomials, whose methods are based on linear algebra constructions. Duan et al. [12] considered the solution of the mixed problem with time fractional-order derivative and the third kind homogeneous boundary condition using integral transform. Moreover, in [13] they also discussed the problem of time fractional diffusion-wave equations on finite interval and obtained variable separated solution using the Laplace transform and its inverse transform. Schneider and Wyss [14] considered the time fractional diffusion and wave equations and derived the corresponding Green functions in closed form for arbitrary space dimensions in terms of Fox functions.

Gorenflo et al. [15] used the similarity method and the method of Laplace transform to obtain the scale-invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. However, an explicit representation of the Green functions for the problem in a half-space is difficult to determine, except in the special cases $\alpha = 1$ (i.e., the first-order time derivative) with arbitrary n, or $n = 1$ with arbitrary α (i.e., the fractional-order time derivative). Huang and Liu [16] considered the time-fractional diffusion equations in a n-dimensional whole-space and half-space. They investigated the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier-Laplace transform. Liu et al. [17] considered time fractional advection dispersion equation and derived the complete solution. The solution of time fractional diffusion-reaction equation was considered by Lu and Liu [18].

As mentioned above, many authors deal with the fractional-order differential equations using integral transforms and the order of derivative was restricted between 0 and 2. The solutions were expressed by convolution form of the Green function and initial value function, which cannot be computed easily. Moreover, sufficient solution regularity is required. In this paper, we consider the Riesz fractional partial differential equations (RFPDE) with periodic functions. The order of derivative can be extended to arbitrary order. Additionally, we only require piecewise smooth of functions about spacial variable and get the analysis solution in the form of series, which can be computed easily.

This paper is organized as follows. Section 2 presents basic definitions and propositions. In Sections 3 and 4, the fundamental solutions of the SRFPDE and the STRFPDE are derived, respectively. The explicit expressions of the fundamental solutions of the SRFPDE and the STRFPDE are obtained.

2. Basic definitions and propositions

In this section, we first present some basic definitions and propositions. We consider the RFPDE as follows:

$$\mathscr{D}_t^{\beta} u(x,t) = D_x^{\alpha} u(x,t), \quad x \in R, \quad t \in R^+, \tag{2.1}$$