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A New Inexactness Criterion for Approximate Logarithmic-Quadratic Proximal Methods

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Abstract. Recently, a class of logarithmic-quadratic proximal (LQP) methods was introduced by Auslender, Teboulle and Ben-Tiba. The inexact versions of these methods solve the sub-problems in each iteration approximately. In this paper, we present a practical inexactness criterion for the inexact version of these methods.

Key words: Variational inequalities; maximal monotone operators; interior proximal methods.

AMS subject classifications: 90C33, 49J40

1 Introduction

Given an operator T, point to set in general, and a closed convex subset C of R^n, the variational inequality problem, denoted by (VI), consists of finding a vector $x^* \in C$ and $g^* \in T(x^*)$ such that

$$(x - x^*)^T g^* \geq 0, \quad \forall x \in C. \tag{1}$$

Our analysis will focus on the case where T is a maximal monotone mapping from R^n into itself and the constraint C is explicitly defined by

$$C := \{x \in R^n : Ax \leq b\},$$

where A is a $p \times n$ matrix, $b \in R^p$ and $p \geq n$. We suppose that the matrix A is of maximal rank, i.e., rank$A = n$ and that int$C = \{x : Ax < b\}$ is nonempty.

It is well known that the VI problem can be alternatively formulated as finding the zero point of a maximal monotone operator $\Pi = T + N_C$, i.e., find $x^* \in C$ such that $0 \in \Pi(x^*)$. A classical method to find the zero point of a maximal monotone operator Π is the proximal point algorithm (e.g., see [5, 8]). For given $x^{k-1} \in R^n$ and $\lambda_k \geq \lambda > 0$, the new iterate x^k is the solution of the following problem:

$$0 \in \Pi(x) + \lambda_k^{-1} \nabla q(x, x^{k-1}), \tag{2}$$

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where

$$q(x, x^{k-1}) = \frac{1}{2}\|x - x^{k-1}\|^2 \tag{3}$$

is a quadratic function of x. The proximal point algorithm can be seen as a regularization method in which the regularization parameter λ_k does not approach $+\infty$, thus avoiding the possible ill behavior of the regularized problems.

The recursion form of (2)-(3) can be written as

$$0 \in x^k - x^{k-1} + \lambda_k \Pi(x^k).$$

However, this ideal form of the method is often impractical, since the exact iteration (2) maybe in many cases require a computation as difficult as solving the original problem $0 \in T(x^*)$. In [8], Rockafellar has given an inexact variant of the method

$$e^k \in x^k - x^{k-1} + \lambda_k \Pi(x^k), \tag{4}$$

where $\{e^k\}$ is regarded as an error sequence. The method is called inexact proximal point algorithm. It was shown that if $e^k \to 0$ quickly enough such that

$$\sum_{k=1}^{+\infty} \|e^k\| < +\infty,$$

then $x^k \to z \in R^n$ with $0 \in \Pi(z)$.

Instead of using the quadratical function (3) as the proximal term, Eckstein [4] investigated the Bregman-function-based proximal method and has proved that the sequence $\{x^k\}$ generated by (4) converges to a root of Π under the following conditions:

$$\sum_{k=1}^{+\infty} \|e^k\| < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty} \langle e^k, x^k \rangle \text{ exists and is finite.} \tag{5}$$

On the other hand, for quadratic proximal method, Han and He [6] have proved the convergence for recursion (4) under the following accuracy criterion

$$\|e^k\| \le \eta_k \|x^k - x^{k-1}\| \quad \text{with} \quad \sum_{k=0}^{+\infty} \eta_k^2 < +\infty. \tag{6}$$

It seems that the accuracy criterion (6) can be checked and complemented in practice more easily than (5).

Recently, Auslender, Teboulle and Ben-Tiba [1] have proposed a new type of proximal interior methods replacing the quadratic function $q(x, x^{k-1})$, by the logarithmic-quadratic function $D(x, x^{k-1})$ (will be specified in Section 3), this method is called logarithmic-quadratic proximal method. In their inexact version[1], they have suggested to use the accuracy criterion of type (5). Since the accuracy criterion of type (6) is more useful in practice, in this paper, we proposed an accuracy criterion of type (6) for the logarithmic-quadratic proximal method.

2 Preliminaries

We list some important results on maximal monotone operator and some basic properties which will be needed in our following analysis. The domain of T and the graph of T are defined by

$$\text{dom}T := \{x | T(x) \neq \emptyset\} \qquad \text{and} \qquad G(T) := \{(x, y) \in R^n \times R^n : y \in T(x)\}.$$