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N DIMENSIONAL FINITE WAVELET FILTERS *1)

Si-long Peng
(*NADEC, Institute of Automation, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, a large class of n dimensional orthogonal and biorthognal wavelet filters (lowpass and highpass) are presented in explicit expression. We also characterize orthogonal filters with linear phase in this case. Some examples are also given, including non separable orhogonal and biorthogonal filters with linear phase.

Key words: n Dimension, Linear phase, Wavelet filters.

1. Introduction

In [1], I. Daubechies constructed orthogonal and biorthogonal wavelet filters in one dimension which have been proved to be very useful in signal and image processing. But except some short filters have explicit solution, almost all the orthogonal filters given in Daubechies' book are numerical results. In some applications, people need filters with high precision, in this case, one need to compute the filters himself. In this paper, we give a class of n dimensional orthogonal and biorthogonal wavelet filters in explicit expression. With these parameterized filters, we can easily realize the adaptive selection of filters in many applications.

Recently, many researchers are working on nonseparable wavelets(see [2], [3], [5], [6] and the references therein) because of the shortcoming of separable filters pointed out in [2]. Using the same method in [5], we can construct n dimensional wavelet filters. It is interesting that among these filters, we can find many nonseparable filters with linear phase, which can not be obtained by using the tensor product of one dimensional wavelet filters.

Our main results and their proofs are proposed in next section. By using the method in section II, some examples are given in section III, including one dimension case and two dimension case with linear phase.

2. Main Results

We will discuss orthogonal in detail first, then using similar method, we give the expression of biorthogonal filters.

2.1 Orthogonal case

The well known method to construct wavelet is MRA. The definition of n dimensional orthogonal MRA is as follows.

Definition. *A sequence of subspaces $\{V_j\}_{j\in \mathbb{Z}}$ of $L^2(R^n)$ is called a MRA if it satisfies the following properties:*

(a). $\cap V_j = \{0\}, \quad \overline{\cup V_j} = L^2(R^n)$;

(b). $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$, *for all* $j \in \mathbb{Z}, x \in R^n$;

(c). There exists a function $\varphi(x) \in V_0$ *such that* $\{\varphi(x-k)\}_{k\in \mathbb{Z}^n}$ *is an orthonormal basis of* V_0.

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Remark.

1). if $x = (x_1, \cdots, x_n)$, then $2x = (2x_1, \cdots, 2x_2)$.

2). $\mathbb{Z}^n$ is the set of all n dimensional integers.

Let $\{V_j\}$ be a n dimensional MRA, then there exists a function $m(\xi)(\xi \in R^n)$ such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi),$$

where $\hat{\varphi}$ is the Fourier transform of φ, and m is called Symbol Function of the scaling function φ. The orthogonality of $\{\varphi(x-k)\}_{k \in \mathbb{Z}^n}$ implies that $m(\xi)$ satisfies

$$\sum_{\nu \in E^n} |m(\xi + \nu\pi)|^2 = 1, \tag{2.1}$$

where E^n denotes the set of all vertexes of n dimensional unit square box.

The construction of φ can be reduced to construct $m(\xi)$. We want to solve (2.1) in some general cases. In this paper, we assume that $m(\xi)$ is a polynomial of $e^{i\xi}$ with the constant term does not equal to 0.

Let $\xi = (\xi_1, \cdots, \xi_n)$. Rewrite $m(\xi)$ in its polyphase form as

$$m(\xi) = \sum_{\nu \in E^n} x^\nu f_\nu(x^2), \tag{2.2}$$

where $x = e^{i\xi}$, $x^\nu = x_1^{\nu_1} \cdot \cdots \cdot x_n^{\nu_n}$, $x_k = e^{i\xi_k}$, $k = 1, \cdots, n$, $\nu = (\nu_1, \cdots, \nu_n) \in E^n$.

It is easy to see that (2.1) is equivalent to

$$\sum_{\nu \in E^n} |f_\nu(e^{i\xi})|^2 = \frac{1}{2^n}. \tag{2.3}$$

To solve (2.3), a theorem is needed as follows.

Theorem 1. *Suppose that $\{f_{e_k}, e_k \in E^n, k = 1, \cdots, 2^n\}$ satisfies (2.3), define*

$$(F_{e_1}, \cdots, F_{e_{2^n}})^T = UD(f_{e_1}, \cdots, f_{e_{2^n}})^T \tag{2.4}$$

where U is any real unitary matrix of size $2^n \times 2^n$, and $D = diag(x^{e_1}, \cdots, x^{e_{2^n}})$, $E^n = \{e_k, k = 1, \cdots, 2^n\}$. Then $\{F_\nu, \nu \in E^n\}$ also satisfy (2.3).

Proof. Since U and D are both unitary matrices, the proof is immediately.

Define

$$\widetilde{m}(\xi) = \sum_{\nu \in E^n} x^\nu F_\nu(x^2), \tag{2.5}$$

then $\widetilde{m}(\xi)$ is a trigonometric polynomial which satisfies (2.1).

Denote the set of all real unitary matrices with size $2^n \times 2^n$ by $\mathcal{U}_n$ and the set of all 2^n dimensional real unit column vectors by $\mathcal{V}_n$. Define

$$(f_{e_1}, \cdots, f_{e_{2^n}})^T = 2^{-\frac{n}{2}}(\otimes_{k=1}^N U_k D_k)V \tag{2.6}$$

where $U_k \in \mathcal{U}_n$, $V \in \mathcal{V}_n$, $D_k = \mathrm{diag}(x^{e_1}, \cdots, x^{e_{2^n}})$, for $k = 1, \cdots, N$.

Let

$$\mathcal{F}_{N,n} = \{f | f = 2^{-\frac{n}{2}}(\otimes_{k=1}^N U_k D_k)V, U_k \in \mathcal{U}_n, k = 1, \cdots, 2^n, V \in \mathcal{V}_n\}. \tag{2.7}$$

Then we have the following theorem:

Theorem 2. *For all $f \in \mathcal{F}_{N,n}$, then the set $\{f_{e_k}, e_k \in E^n, k = 1, \cdots, 2^n\}$ satisfies equation (2.3).*

Proof. The proof is immediately.

Denote $X_E = (x^{e_1}, \cdots, x^{e_{2^n}})$.

Define

$$m(\xi) = X_E \cdot f(x^2) \tag{2.8}$$

where $f \in \mathcal{F}_{N,n}$, $\cdot$ is the matrix multiply operator. Then $m(\xi)$ satisfies (2.1).