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A MODIFIED LEVENBERG-MARQUARDT ALGORITHM FOR SINGULAR SYSTEM OF NONLINEAR EQUATIONS *1)

Jin-yan Fan
(*Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China*)

Abstract

Based on the work of paper [1], we propose a modified Levenberg-Marquardt algorithm for solving singular system of nonlinear equations $F(x) = 0$, where $F(x) : R^n \to R^n$ is continuously differentiable and $F'(x)$ is Lipschitz continuous. The algorithm is equivalent to a trust region algorithm in some sense, and the global convergence result is given. The sequence generated by the algorithm converges to the solution quadratically, if $||F(x)||_2$ provides a local error bound for the system of nonlinear equations. Numerical results show that the algorithm performs well.

Key words: Singular nonlinear equations, Levenberg-Marquardt method, Trust region algorithm, Quadratic convergence.

1. Introduction

We consider the problem for solving the system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where $F(x) : R^n \to R^n$ is continuously differentiable and $F'(x)$ is Lipschitz continuous. Throughout the paper, we assume that the solution set of (1.1) is nonempty and denoted by X^*. And in all cases $||\cdot||$ refers to the 2-norm.

The Levenberg-Marquardt method (see [2, 3]) for nonlinear equations (1.1) computes the trial step by

$$d_k = -(J(x_k)^T J(x_k) + \mu_k I)^{-1} J(x_k)^T F(x_k), \tag{1.2}$$

where $J(x_k) = F'(x_k)$ is the Jocabi, and $\mu_k \geq 0$ is a parameter being updated from iteration to iteration. The Levenberg-Marquardt step (1.2) is a modification of the Newton's step. The parameter μ_k is introduced to overcome the difficulties caused by singularity or near singularity of $J(x_k)$.

There are various choices of the parameter μ_k in (1.2). Recently, paper [11] shows, if the parameter is chosen as $\mu_k = ||F(x_k)||^2$, and if the initial point is sufficiently close to x^*, then, under a weaker condition than nonsingularity that $||F(x)||$ provides a local error bound near the solution, the Levenberg-Marquardt method has a quadratic rate of convergence. Paper [1] extends the result in [11], and obtains that the quadratic convergence still holds if the parameter is chosen as $\mu_k = ||F(x_k)||$. Although the numerical results show that the choice of $\mu_k = ||F(x_k)||$ performs better than that of $\mu_k = ||F(x_k)||^2$ [1], it does not perform very well when the sequence is far away from the solution. In this paper, based on the work of papers [1] and [11], we consider a modified Levenberg-Marquardt method, in which the parameter is chosen as $\mu_k ||F(x_k)||$ with μ_k being updated by trust region techniques.

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Definition 1.1. *Let N be a subset of R^n such that $N \cap X^* \neq \emptyset$. We say that $||F(x)||$ provides a local error bound on N for system (1.1), if there exists a positive constant $c > 0$ such that*

$$||F(x)|| \geq c dist(x, X^*), \qquad \forall x \in N.$$

Note that, if $J(x^*)$ is nonsingular at a solution x^* of (1.1), then x^* is an isolated solution, hence $||F(x)||$ provides a local error bound on some neighbourhood of x^*. However, the converse is not necessarily true, see example in [11]. Thus, a local error bound condition is weaker than nonsingularity.

In the next section, we present the modified algorithm, and show that it is equivalent to a trust region algorithm. In section 3, global convergence result is proved. The algorithm converges to a stationary point if a trial step is accepted only when the actual reduction of the function is at least a fraction of the predicted reduction (the reduction in the approximation model). In section 4, local convergence analyses are made. It is shown that the algorithm converges quadratically if $||F(x)||$ provides a local error bound condition near the solution. Finally in section 5, we present the numerical results for some singular systems of nonlinear equations.

2. Modified Levenberg-Marquardt Algorithm and Trust Region

In this section, we first present the general trust region algorithm, then present our new Levenberg-Marquardt algorithm. The relationship between these two algorithms is given.

At the beginning of each iteration in a general trust region algorithm for nonlinear equations, a trial step d_k is computed by solving the subproblem:

$$\begin{aligned} &\min_{d \in R^n} \ ||F_k + J_k d||^2 \triangleq \varphi_k(d) \\ &s.\ t. \ \ ||d|| \leq \Delta_k, \end{aligned} \tag{2.1}$$

where $F_k = F(x_k), J_k = J(x_k)$, and $\Delta_k > 0$ is the current trust region bound. The actual reduction and the predicted reduction of the function are defined as follows:

$$Ared_k = ||F_k||^2 - ||F(x_k + d_k)||^2,$$

and

$$Pred_k = \varphi_k(0) - \varphi_k(d_k).$$

The ratio between these two reductions is defined by

$$r_k = \frac{Ared_k}{Pred_k},$$

which is used to decide whether the trial step is acceptable and to adjust the new parameter μ_k. Paper [12] presents a class of trust region algorithms for nonlinear equations in any arbitrary norm, and gives the global and local convergence results. The general trust region algorithm for nonlinear equations in 2-norm can be stated as follows :

Algorithm 2.1. *(Trust region algorithm for nonlinear equations)*

Step 1. Given $x_1 \in R^n, \Delta_1 > 0, \varepsilon \geq 0, 0 \leq p_0 \leq p_1 \leq p_2 < 1, k := 1.$

Step 2. If $||J_k^T F_k|| \leq \varepsilon$, then stop;
Solve (2.1) giving d_k.

Step 3. Compute $r_k = Ared_k / Pred_k$;