Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0147

Vol. **23**, No. 2, pp. 476-499
February 2018

An Enriched Multiscale Mortar Space for High Contrast Flow Problems

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Received 2 September 2016; Accepted (in revised version) 15 May 2017

Abstract. Mortar methods are widely used techniques for discretizations of partial differential equations and preconditioners for the algebraic systems resulting from the discretizations. For problems with high contrast and multiple scales, the standard mortar spaces are not robust, and some enrichments are necessary in order to obtain an efficient and robust mortar space. In this paper, we consider a class of flow problems in high contrast heterogeneous media, and develop a systematic approach to obtain an enriched multiscale mortar space. Our approach is based on the constructions of local multiscale basis functions. The multiscale basis functions are constructed from local problems by following the framework of the Generalized Multiscale Finite Element Method (GMsFEM). In particular, we first create a local snapshot space. Then we select the dominated modes within the snapshot space using an appropriate Proper Orthogonal Decomposition (POD) technique. These multiscale basis functions show better accuracy than polynomial basis for multiscale problems. Using the proposed multiscale mortar space, we will construct a multiscale finite element method to solve the flow problem on a coarse grid and a preconditioning technique for the fine scale discretization of the flow problem. In particular, we develop a multiscale mortar mixed finite element method using the mortar space. In addition, we will design a two-level *additive* preconditioner and a two-level *hybrid* preconditioner based on the proposed mortar space for the iterative method applied to the fine scale discretization of the flow problem. We present several numerical examples to demonstrate the efficiency and robustness of our proposed mortar space with respect to both the coarse multiscale solver and the preconditioners.

AMS subject classifications: 65M99, 65M55, 65M60

Key words: Multiscale, domain decomposition, finite elements.

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1 Introduction

In this paper, we consider the following second order elliptic differential equation in mixed form:

$$\boldsymbol{q}+\kappa\nabla u=0 \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{q}=f \qquad \text{in } \Omega, \tag{1.1b}$$

$$u=0 \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where $\Omega\subset\mathbb{R}^d$ ($d=2,3$) is a bounded polyhedral domain with outward unit normal vector $\boldsymbol{n}$ on the boundary, $f\in L^2(\Omega)$, κ represents the permeability field that varies over multiple spacial scales. Possible applications of (1.1a)-(1.1c) include flows in porous media, diffusion and transport of passive chemicals or heat transfer in heterogeneous media. Solving (1.1a)-(1.1c) can be challenging if Ω is large and the permeability κ is heterogeneous with multiple scales and high contrast, which is a common characteristic in many industrial, scientific, engineering, and environmental applications. Direct simulation requires very fine meshes and this makes the corresponding algebraic system very large and ill conditioned (due to both the small mesh size and the high contrast of the coefficient). Thus direct simulation is computationally intractable.

In order to solve (1.1a)-(1.1c) efficiently, various reduced-order methods have been proposed and applied. These methods include numerical upscaling (see, e.g., [1,2]), variational multiscale method (see, e.g., [3, 4]), multiscale finite element method (see, e.g., [5–8]), mixed multiscale finite element methods (see, e.g., [9,10]), the multiscale finite volume method (see, e.g., [11]), mortar multiscale finite element method (see, e.g., [12–15]), multiscale hybrid-mixed finite element methods (see, e.g., [16, 17]), generalized multiscale finite element methods (see, e.g., [18–22]) and weak Galerkin generalized multiscale finite element method [23]. These methods typically use some type of global couplings in the coarse grid level to link the sub-grid variations of neighboring coarse regions. We will, in this paper, consider the global coupling via the mortar framework. The mortar framework offers many advantages, such as the flexibility in the constructions of the coarse grid and sub-grid capturing tools. The framework also gives a smaller dimensional global system since the degrees of freedom are reduced to coarse region boundaries. The connectivity of the sub-grid variations is typically enforced using a Lagrange multiplier. For multiscale problems, the choice of the mortar space for the Lagrange multiplier requires a very careful construction, in order to obtain an efficient and robust method. To construct an accurate mortar space with a small dimension, we will apply the recently developed GMsFEM, which offers a systematic approach for model reduction. In particular, we first create a local snapshot space for every coarse edge. We obtain this space by first solving some local problems on a small region containing an edge, and then restricting the solutions to the edge. Next, we select the dominated modes within the snapshot space using an appropriate POD technique. These dominated modes form the basis for the mortar space. We will apply our mortar space in two related formu-