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A Fifth-Order Combined Compact Difference Scheme for Stokes Flow on Polar Geometries

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Abstract. Incompressible flows with zero Reynolds number can be modeled by the Stokes equations. When numerically solving the Stokes flow in stream-vorticity formulation with high-order accuracy, it will be important to solve both the stream function and velocity components with the high-order accuracy simultaneously. In this work, we will develop a fifth-order spectral/combined compact difference (CCD) method for the Stokes equation in stream-vorticity formulation on the polar geometries, including a unit disk and an annular domain. We first use the truncated Fourier series to derive a coupled system of singular ordinary differential equations for the Fourier coefficients, then use a shifted grid to handle the coordinate singularity without pole condition. More importantly, a three-point CCD scheme is developed to solve the obtained system of differential equations. Numerical results are presented to show that the proposed spectral/CCD method can obtain all physical quantities in the Stokes flow, including the stream function and vorticity function as well as all velocity components, with fifth-order accuracy, which is much more accurate and efficient than low-order methods in the literature.

AMS subject classifications: 65N06

Key words: Stokes flow, combined compact difference (CCD) scheme, truncated Fourier series, shifted grid, coordinate singularity.

1. Introduction

The two-dimensional (2D) Stokes equations in the usual velocity-pressure formulation are given as follows,

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$$-\Delta u + \frac{\partial p}{\partial x} = f_1, \quad \text{in} \quad \Omega, \tag{1.1}$$

$$-\Delta v + \frac{\partial p}{\partial y} = f_2, \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{in} \quad \Omega, \tag{1.3}$$

with boundary conditions

$$u = u_1(x,y), \quad v = v_1(x,y), \quad \text{on} \quad \Gamma = \partial\Omega, \tag{1.4}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the well-known Laplacian operator, (u,v) is the velocity field, p is the pressure, (f_1, f_2) is the external force, Ω is the 2D domain, and Γ is its boundary.

A flow governed by the Stokes equations is known as a creeping flow or Stokes flow. Such flow is used for the fluid with very low Reynolds number, that is the inertia is very small compared to the viscous force. When the inertial term is neglected, i.e., Reynolds number is zero, 2D Navier-Stokes equations reduce to Stokes equations (1.1)-(1.3). We refer [26] for a brief introduction to the life at low Reynolds number, and refer [13] for discussions on various methods for solving the viscous incompressible flow at low Reynolds numbers. The above 2D Stokes problem (1.1)-(1.4) in rectangular domain was numerically studied in [12] by a fourth-order compact MAC finite difference scheme. However, in many physical problems, one often needs to solve flow equations in a non-Cartesian domain, such as polar or cylindrical domains. For example, Navier-Stokes equations in the polar or cylindrical domains are discussed in [2, 10, 16, 19, 20, 24, 28, 29, 34, 36, 37]. This paper will focus on the 2D Stokes equations on polar geometries, including a unit disk and an annulus domain.

Now we first introduce the stream function ψ [27, 35], which satisfies

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}. \tag{1.5}$$

Through a simple calculation, it is easy to show that the above Stokes equations (1.1)-(1.3) is equivalent to the following boundary value problem of biharmonic equation for ψ,

$$\Delta^2 \psi = f(x,y), \quad \text{in } \Omega, \tag{1.6}$$

where

$$f = \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x},$$

and the boundary conditions (1.4) can be transformed as

$$\psi = g(x,y), \quad \frac{\partial \psi}{\partial \vec{n}} = h(x,y), \quad \text{on } \Gamma = \partial\Omega, \tag{1.7}$$

where

$$h(x,y) = v_1 n_1 - u_1 n_2, \quad g(x,y) = \int_\Gamma -u_1 dy + v_1 dx + g(x_0, y_0).$$