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A Modified Relaxed Positive-Semidefinite and Skew-Hermitian Splitting Preconditioner for Generalized Saddle Point Problems

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Abstract. Based on the relaxed factorization techniques studied recently and the idea of the simple-like preconditioner, a modified relaxed positive-semidefinite and skew-Hermitian splitting (MRPSS) preconditioner is proposed for generalized saddle point problems. Some properties, including the eigenvalue distribution, the eigenvector distribution and the minimal polynomial of the preconditioned matrix are studied. Numerical examples arising from the mixed finite element discretization of the Oseen equation are illustrated to show the efficiency of the new preconditioner.

AMS subject classifications: 65F10, 65F50

Key words: Generalized saddle point problems, positive-semidefinite and skew-Hermitian splitting, preconditioning, Krylov subspace method.

1. Introduction

Recently, a large amount of work has been devoted to the problem of solving large linear systems in saddle point form. Such systems arise in a wide variety of scientific computing and engineering applications, such as geomechanics [15], mixed finite element approximation of elliptic partial differential equations [16], piezoelectric structures [17], meshfree approximation of elastic mechanics [21], computational fluid dynamics [22], optimization problems [25] and so on. For more background information on the applications of saddle point problems, please see [1, 13, 29] and references therein.

This work is concerned with the iterative solution of the following large sparse generalized saddle point linear system

$$\mathscr{A}\mathrm{x} \equiv \begin{bmatrix} A & B^* \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1.1}$$

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where $A \in \mathbb{C}^{n\times n}$ is non-Hermitian positive-definite (i.e., its Hermitian part $\frac{1}{2}(A+A^*)$ is positive-definite), $B \in \mathbb{C}^{m\times n}$ $(m \leq n)$ is a rectangular matrix of full row rank, B^* is the conjugate transpose of B, $C \in \mathbb{C}^{m\times m}$ is Hermitian positive-semidefinite, $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$ are two given vectors. In particular, when $A \in \mathbb{C}^{n\times n}$ is Hermitian positive-definite and $C = 0$, the linear system (1.1) is often called the standard saddle point problem [13]. The above assumptions ensure that the block two-by-two matrix $\mathscr{A}$ is nonsingular [13, Theorem 3.4]. For the nonsingularity of a general block two-by-two matrix, please see [4, Lemma 2.1]. Thus, the solution of (1.1) exists and is unique.

In many cases, the matrices A, B and C are large sparse and iterative techniques are preferable for solving (1.1). Since the generalized saddle point matrix $\mathscr{A}$ is non-Hermitian positive-semidefinite and often ill-conditioned, preconditioning is in most cases indispensable for iterative solution of (1.1) [29]. Let

$$\mathscr{A} = \begin{bmatrix} A & B^* \\ -B & C \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} S & B^* \\ -B & 0 \end{bmatrix} = \hat{\mathscr{H}} + \hat{\mathscr{S}}$$

be the splitting of $\mathscr{A}$ into its Hermitian and skew-Hermitian parts, where $H = \frac{1}{2}(A+A^*)$ and $S = \frac{1}{2}(A-A^*)$ are the Hermitian part and the skew-Hermitian part of the (1,1) block matrix A, respectively. Applying the Hermitian and skew-Hermitian splitting (HSS) iteration method

$$\begin{cases} (\alpha I + \hat{\mathscr{H}})\mathrm{x}^{k+\frac{1}{2}} = (\alpha I - \hat{\mathscr{S}})\mathrm{x}^k + b, \\ (\alpha I + \hat{\mathscr{S}})\mathrm{x}^{k+1} = (\alpha I - \hat{\mathscr{H}})\mathrm{x}^{k+\frac{1}{2}} + b, \end{cases} \qquad (k = 0, 1, 2, \cdots.) \tag{1.2}$$

proposed by Bai, Golub and Ng in [8], Benzi and Golub constructed a class of HSS preconditioners

$$\hat{\mathscr{P}}_{HSS} = \frac{1}{2\alpha} \begin{bmatrix} \alpha I + H & 0 \\ 0 & \alpha I + C \end{bmatrix} \begin{bmatrix} \alpha I + S & B^* \\ -B & \alpha I \end{bmatrix} \tag{1.3}$$

for generalized saddle point problems (1.1), where α is a given positive parameter and I is the identity matrix with suitable dimensions. The HSS iteration method is a very promising method since it is convergent unconditionally for solving non-Hermitian positive-definite linear systems [8]. In addition, the unconditional convergence property can be extended to the generalized saddle point problems [12] and the general non-Hermitian positive-semidefinite linear systems [6]. As a preconditioner, the pre-factor has no effect on the preconditioned system. So, in many cases, we can use the following one

$$\begin{aligned} \mathscr{P}_{HSS} &= \frac{1}{\alpha} \begin{bmatrix} \alpha I + H & 0 \\ 0 & \alpha I + C \end{bmatrix} \begin{bmatrix} \alpha I + S & B^* \\ -B & \alpha I \end{bmatrix} \\ &= \begin{bmatrix} \alpha I + A + \frac{1}{\alpha} HS & B^* + \frac{1}{\alpha} HB^* \\ -B - \frac{1}{\alpha} CB & \alpha I + C \end{bmatrix} \end{aligned} \tag{1.4}$$

to replace the original HSS preconditioner (1.3). Although $\mathscr{P}_{HSS}$ no longer relates to an alternating direction iteration method (1.2), but it is of no consequence when $\mathscr{P}_{HSS}$ is used as a preconditioner for the Krylov subspace method like GMRES [18, 21]. To improve the preconditioning effects of the HSS preconditioner and accelerate the convergence rate of