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Computing Solutions of the Yang-Baxter-like Matrix Equation for Diagonalisable Matrices

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Abstract. The Yang-Baxter-like matrix equation $AXA = XAX$ is reconsidered, where A is any complex square matrix. A collection of spectral solutions for the unknown square matrix X were previously found. When A is diagonalisable, by applying the mean ergodic theorem we propose numerical methods to calculate those solutions.

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Key words: Matrix equation, mean ergodic theorem, diagonalisable matrix.

1. Introduction

In this article, we reconsider the matrix equation

$$AXA = XAX \ , \tag{1.1}$$

where both A and X are constant complex square matrices of the same size $(n \times n)$. Eq. (1.1) has been called *Yang-Baxter-like*, after the Yang-Baxter equation for two-dimensional integrable models in statistical mechanics [1,13]. The *parameter-dependent* equation

$$A(u)B(u+v)A(v) = B(u)A(u+v)B(v), \tag{1.2}$$

where A and B are rational matrix functions of their arguments, obviously reduces to Eq. (1.1) when A and B are constant matrices. The size of A and B in applied problems is typically not large — e.g. the matrices that were considered in Refs. [1,13] are only 4×4, namely

$$\begin{bmatrix} 1+u & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & 1+u \end{bmatrix} \text{ and } \begin{bmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{bmatrix},$$

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respectively. The Yang-Baxter equation has been applied for decades by physicists and mathematicians in many areas such as group theory, braid groups and knot theory [7, 10, 14]. In contrast, the Yang-Baxter-like matrix equation (1.1) has not attracted much attention from matrix theorists, perhaps due to its nonlinearity or their lack of background in knot theory and braid groups (or quantum mechanics). Even for matrices of small size, it has been difficult to find all of the solutions. Some solutions have been obtained, but mostly via direct computation from the polynomial equations corresponding to multiplying out the matrix equation in specific cases. There is still no systematic approach to the existence and computation of solutions of Eq. (1.1) in general, but the numerical method proposed here yields spectral solutions for any matrix size.

The concept of braids was introduced in 1925 by Emil Artin, and a braid group with n strands B_n is a group where the multiplication of a braid s to another braid t corresponds to gluing s onto the bottom of t. It follows that every braid s has a unique inverse braid t ($st = ts = e$, where e is the unit braid such that strands are preserved. It is fundamental that there are elementary braids $s_1, s_2, \cdots, s_{n-1}$, where the i-th strand of s_i goes over to the right of the $(i+1)$-th strand of s_i for each i, to generate the whole braid group B_n. Furthermore, these braids satisfy the Yang-Baxter-like relation

$$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i \tag{1.3}$$

for each $i = 1, \cdots, n-2$, and also $s_is_j = s_js_i$ for any i and j where $|i-j| > 1$. This relation (1.3) evidently has the same form as the matrix equations (1.1) and (1.2). The matrix equations may also be viewed as word equations — cf. Ref. [8] and references therein for more detail. Given a uniquely divisible group G where every element has an n-th root for any positive integer n, a word equation has the form $W(X,A) = B$, where W is a finite word consisting of the unknown element X and the known element A of G and B is a given element in G. Under certain conditions, a solution can be obtained in terms of radicals. Eq. (1.1) written as $W(X,A) = AXA - XAX = 0$ is a word equation. However, since the class of all square matrices is not a group under matrix multiplication, the general setting of word equations does not apply unless we restrict our consideration to invertible matrices of the same size say. (Even then, not every invertible matrix has a root, as is often required in solution techniques for solving word equations.) Thus computing solutions of the Yang-Baxter-like equation (1.1) in practice is generally a challenging problem, and different techniques should be employed.

In first considering the existence and computation of solutions to the Yang-Baxter-like matrix equation (1.1) of arbitrary size, we obtained some numerical solutions when A is a nonsingular quasi-stochastic matrix such that A^{-1} is stochastic [3]. Recently, we have proven some general existence results for an arbitrary square matrix A, by finding a collection of solutions of Eq. (1.1) in terms of spectral projections associated with all of its eigenvalues [4]. More solutions were found in Ref. [5] for some classes of matrix A with special Jordan canonical forms, based on a general result for commuting solutions, but there has not been any actual numerical computation of such spectral solutions. In this article, we show that the solutions found in Ref. [4] can be computed by means of the mean ergodic theorem if A is diagonalisable.