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The Mediating Morphism of the Multilinear Optimal Map

Seak-Weng Vong[1,*], Xiao-Qing Jin[1] and Jin-Hua Wang[2]

[1] *Department of Mathematics, University of Macau, Macao, China.*
[2] *Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou, China.*

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Abstract. In this short note, we study a relation between the tensor product of matrices and a multilinear map defined by the optimal operator. In this particular case, the linear transform (mediating morphism) hidden in the abstract definition of the general tensor product can be determined explicitly.

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1. Introduction

Many preconditioners have been proposed in structured matrix computations since 1986 [13]. Among them, the most famous ones are Strang's circulant preconditioner [13], the optimal preconditioner [3] and the superoptimal preconditioner [14]. In this note, we concentrate on the study of a multilinear operator defined by the optimal preconditioner. Thus given a unitary matrix $U \in \mathbb{C}^{n\times n}$, let

$$\mathscr{M}_U \equiv \left\{U^*\Lambda U \mid \Lambda \text{ is any } n\times n \text{ diagonal matrix}\right\}. \tag{1.1}$$

For an arbitrary matrix $A \in \mathbb{C}^{n\times n}$, the optimal preconditioner $c_U(A)$ is defined to be the solution of

$$\min_{W\in\mathscr{M}_U} \|A-W\|_F\,, \tag{1.2}$$

where $\|\cdot\|_F$ is the Frobenius norm and W runs over $\mathscr{M}_U$ [1,3]. Computational and mathematical properties of the optimal preconditioner $c_U(A)$ have been studied extensively [1,2,4,5,11], and it has also been considered from an operator viewpoint [8,10].

*Corresponding author. *Email addresses:* `swvong@umac.mo` (S.-W. Vong), `xqjin@umac.mo` (X.-Q. Jin), `wjh@zjut.edu.cn` (J.-H. Wang)

In this short note, we study a relation between a multilinear map f defined by the operator c_U and the tensor matrix product $\otimes$, and seek an exact form of the mediating morphism g such that $f = g \circ \otimes$. Matrices with tensor structure can be solved efficiently by the optimal preconditioner [9], and such matrices have many practical applications — e.g. see [6] for an application to the inverse heat problem. We believe the result in this paper may give some insights for designing preconditioners for these matrices. Some preliminaries related to the concepts involved are reviewed in the next section, and our main results are given in the subsequent section.

2. Preliminaries

Some important properties of the optimal preconditioner $c_U(A)$ defined by (1.2) are first summarised. We use $\delta(A)$ to denote the diagonal matrix with diagonal the same as the diagonal of the matrix A — i.e. if $A = (a_{pq})$, then

$$\delta(A) = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} .$$

The following result can be found in Refs. [1,8,10].

Theorem 2.1. *For arbitrary $A = (a_{pq}) \in \mathbb{C}^{n\times n}$, the optimal preconditioner $c_U(A)$ is uniquely determined by A and given by*

$$c_U(A) = U^*\delta(UAU^*)U . \tag{2.1}$$

Proof. For the completeness of this note, we include the following brief proof. Noting that the Frobenius norm is unitary invariant,

$$\|W - A\|_F = \|U^*\Lambda U - A\|_F = \|\Lambda - UAU^*\|_F .$$

Since Λ can only affect the diagonal entries of UAU^*, the minimizer of $\|\Lambda - UAU^*\|_F$ over all diagonal matrices is $\Lambda = \delta(UAU^*)$, so $c_U(A) = U^*\delta(UAU^*)U$. $\square$

Suppose now that the Banach algebra of all $n \times n$ matrices over the complex field is equipped with a matrix norm $\|\cdot\|$ and denoted by $(\mathbb{C}^{n\times n}, \|\cdot\|)$; and let $(\mathscr{M}_U, \|\cdot\|)$ be the sub-algebra of $(\mathbb{C}^{n\times n}, \|\cdot\|)$, where $\mathscr{M}_U$ is defined by (1.1). Then obviously, c_U is a linear operator from $(\mathbb{C}^{n\times n}, \|\cdot\|)$ into $(\mathscr{M}_U, \|\cdot\|)$. We call c_U the optimal operator, and there is the following theorem on properties involving the operator norms of c_U — cf. Refs. [1,8]):

Theorem 2.2. *We have*

(i) $\|c_U\|_F \equiv \sup\limits_{\|A\|_F=1} \|c_U(A)\|_F = 1$; *and*

(ii) $\|c_U\|_2 \equiv \sup\limits_{\|A\|_2=1} \|c_U(A)\|_2 = 1$, *where $\|\cdot\|_2$ is the l_2 norm of the matrix.*