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A Block Matrix Loop Algebra and Bi-Integrable Couplings of the Dirac Equations

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Abstract. A non-semisimple matrix loop algebra is presented, and a class of zero curvature equations over this loop algebra is used to generate bi-integrable couplings. An illustrative example is made for the Dirac soliton hierarchy. Associated variational identities yield bi-Hamiltonian structures of the resulting bi-integrable couplings, such that the hierarchy of bi-integrable couplings possesses infinitely many commuting symmetries and conserved functionals.

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1. Introduction

Zero curvature equations on semi-direct sums of loop algebras generate integrable couplings [1, 2], and the associated variational identities [3, 4] furnish Hamiltonian structures and bi-Hamiltonian structures of the resulting integrable couplings [5–7]. It is an important step in generating Hamiltonian structures to search for non-degenerate, symmetric and ad-invariant bilinear forms on the underlying loop algebras [8, 9]. Special semi-direct sums of loop algebras bring various interesting integrable couplings [8–12], including higher dimensional local bi-Hamiltonian ones [13–16] that greatly enrich multi-component integrable systems.

A zero curvature representation of a system of form

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \cdots), \tag{1.1}$$

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where u is a column vector of dependent variables, means there exists a Lax pair [17] $U=U(u,\lambda)$ and $V=V(u,\lambda)$ belonging to a matrix loop algebra such that

$$U_t - V_x + [U,V] = 0 \tag{1.2}$$

generates the system [18]. An integrable coupling of the system (1.1) is an integrable system of the following form [13, 14]:

$$\bar{u}_t = \bar{K}_1(\bar{u}) = \begin{bmatrix} K(u) \\ S(u,u_1) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \end{bmatrix}, \tag{1.3}$$

where u_1 is a new column vector of dependent variables. Further, an integrable coupling (1.3) is called nonlinear if the supplementary sub-vector field $S(u,u_1)$ is nonlinear with respect to the sub-vector u_1 [19, 20], and an integrable system of the form

$$\bar{u}_t = \bar{K}(\bar{u}) = \begin{bmatrix} K(u) \\ S_1(u,u_1) \\ S_2(u,u_1,u_2) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}, \tag{1.4}$$

is called a bi-integrable coupling of (1.1). In (1.4), it is notable that S_2 depends on the second column sub-vector u_2 but S_1 does not. We now proceed to use zero curvature equations in order to explore the generation of bi-integrable couplings and Hamiltonian structures for the resulting integrable couplings, through variational identities associated with enlarged Lax pairs.

One class of important integrable couplings consists of the so-called dark equations, motivated by the mysterious dark energy and dark matter envisaged in astronomy and cosmology [21]:

$$\begin{cases} u_t = K(x,t,u,u_x,u_{xx},\cdots), \\ \psi_t = A(u,\partial_x)\psi, \end{cases} \tag{1.5}$$

where $A(u,\partial_x)$ ia a linear differential operator. Dark energy is an hypothetical form of energy that is said to permeate all space, proposed to account for a missing part of the total mass in the entire universe (not in the form of visible stars and planets) in order to explain the observed increased rate of expansion of the universe [22, 23]. Dark equations are linear extensions of the original system, which can extend the original equation further and further like general integrable couplings do. Importantly, they represent the large majority of integrable equations that we can readily consider, particularly in the study of integrable systems with multi-components, whereas nonlinear extensions are less amenable. In theory, they generalise the symmetry problem, and the first-order perturbation equations are special examples of dark equations with solutions that solve the original physical models to higher precision [13].

A soliton hierarchy is usually associated with a spectral problem

$$\phi_x = U\phi, \qquad U = U(u,\lambda) \in \tilde{\mathfrak{g}}, \tag{1.6}$$