East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.130713.221113a

Vol. **3**, No. 4, pp. 352-362
November 2013

Further Solutions of a Yang-Baxter-like Matrix Equation

Jiu Ding[1], Chenhua Zhang[1,*] and Noah H. Rhee[2]

[1] *Department of Mathematics, The University of Southern Mississippi, Hattiesburg, MS 39406-5045, USA.*
[2] *Department of Mathematics and Statistics, University of Missouri - Kansas City, Kansas City, MO 64110-2499, USA.*

Received 13 July 2013; Accepted (in revised version) 22 November 2013

Available online 28 November 2013

Abstract. The Yang-Baxter-like matrix equation $AXA = XAX$ is reconsidered, and an infinite number of solutions that commute with any given complex square matrix A are found. Our results here are based on the fact that the matrix A can be replaced with its Jordan canonical form. We also discuss the explicit structure of the solutions obtained.

AMS subject classifications: 15A18
Key words: Matrix equation, Jordan canonical form, projector.

1. Introduction

Let A be a complex $n \times n$ matrix in the quadratic matrix equation

$$AXA = XAX \tag{1.1}$$

for the unknown matrix $X \in \mathbf{C}^{n\times n}$, which we refer to as a *Yang-Baxter-like matrix equation* since its form is similar to the classic parameter-free Yang-Baxter equation originally introduced by Yang in 1967 [9] and then independently by Baxter five years later [2] in the field of statistical mechanics. The Yang-Baxter equation is also closely related to several mathematical areas such as braid groups and knot theory (e.g. see Refs. [8,10] for more details and related topics), and some other applications have already appeared — e.g. see Refs. [1,7].

Obviously, Eq. (1.1) has the two trivial solutions $X = 0$ and $X = A$, but its nonlinearity makes it difficult to find solutions in general. Recently, several classes of solutions of Eq. (1.1) have been obtained for some special cases of the given matrix A. When A is a nonsingular quasi-stochastic matrix such that A^{-1} is a stochastic matrix, the Brouwer fixed

*Corresponding author. *Email addresses:* `Jiu.Ding@usm.edu` (Jiu Ding), `Chenhua.Zhang@usm.edu` (Chenhua Zhang), `RheeN@umkc.edu` (Noah H. Rhee)

point theorem was used to prove that a solution exists and some numerical solutions were obtained via direct iteration [5]. When A is a projector or idempotent matrix (i.e. such that $A^2 = A$), all of the solutions of Eq. (1.1) have been found [3]. With the help of the spectral projection theorem in the analytic theory of matrices, a general spectral solution result of Eq. (1.1) was obtained, without any hypothesis about the given matrix A [6]. In particular, generalised eigenspaces and the concept of the index of an eigenvalue were used to explore the analytic properties of the given matrix A.

In this article, we continue our investigation of Eq. (1.1) to find further nontrivial solutions for a given matrix A. Until now, only finitely many solutions have been obtained, except when A is a projector. We recall that the centraliser of A, consisting of all of the solutions of the linear matrix equation $AX = XA$, is an n-dimensional subspace of $C^{n\times n}$ — cf. Theorem 5.16 of Ref. [4]. Similarly, except for the trivial case of $n = 1$ the general Yang-Baxter-like matrix equation has infinitely many solutions represented by a system of n^2 quadratic equations with n^2 unknowns, which constitute a sub-manifold of $\mathbf{C}^{n\times n}$. For instance, directly solving the 2×2 form of Eq. (1.1) with a simple Jordan block

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

gives the nontrivial solutions

$$B = \begin{bmatrix} b & (b-1)^2 \\ -1 & 2-b \end{bmatrix},$$

where b is any complex number. Unlike in Ref. [6], where only a finite collection of solutions were found, we explore the solution set structure for some cases by finding an expression for infinitely many solutions. Our approach is based on several simple results and a format reduction where A in Eq. (1.1) is replaced by its Jordan canonical form, which simplifies the computation.

In Section 2, we present simple sufficient conditions under which a square matrix B is a solution to Eq. (1.1). In Section 3, we give the explicit expression of the solutions to Eq. (1.1) for several types of Jordan canonical form for A. A numerical example that contrasts our results here with those of Ref. [6] is discussed in Section 4, and our conclusions are in Section 5.

2. Sufficient Conditions for a Solution

Throughout this article, we assume that A is a fixed square $n \times n$ matrix. The set $\sigma(A)$ denoting all the eigenvalues of A is called the *spectrum* of A. An eigenvalue is said to be *semi-simple* if its algebraic multiplicity and geometric multiplicity are equal; and if both of these multiplicities equal one, the eigenvalue is called *simple*. An eigenvalue that is not semi-simple is called a *defective* eigenvalue. We now present several general sufficient conditions for solving Eq. (1.1), which are to be applied in the next section for structural analysis of the solutions for several particular matrices A.