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L-Factors and Adjacent Vertex-Distinguishing Edge-Weighting

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Abstract. An edge-weighting problem of a graph G is an assignment of an integer weight to each edge e. Based on an edge-weighting problem, several types of vertex-coloring problems are put forward. A simple observation illuminates that the edge-weighting problem has a close relationship with special factors of the graphs. In this paper, we generalise several earlier results on the existence of factors with pre-specified degrees and hence investigate the edge-weighting problem — and in particular, we prove that every 4-colorable graph admits a vertex-coloring 4-edge-weighting.

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1. Introduction

In this paper, we consider only finite, undirected and simple graphs. For a graph $G = (V,E)$, if $v \in V(G)$ and $e \in E(G)$ let $v \sim e$ mean that v is an end-vertex of e. For $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v. For a spanning subgraph H of G and $W \subseteq V(G)$, we use $d_H(v)$ for the number of neighbors of v in H and $d_W(v) = |N_G(v) \cap W|$. In addition, let $\omega(H)$ denote the number of connected components of H. A *k-vertex coloring* c of G is an assignment of k integers, $1, 2, \cdots, k$, to the vertices of G, and the color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two adjacent vertices share the same color. A graph G is *k-colorable* if G has a proper k-vertex coloring. The *chromatic number* $\chi(G)$ is the minimum number r such that G is r-colorable. For integers a and b, $[a, b]$ denotes the integers n with $a \le n \le b$. Notations and terminologies that are not defined here may be found in Ref. [6].

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A *k-edge-weighting* of a graph G is an assignment $w : E(G) \to \{1, \cdots, k\}$. An edge-weighting naturally induces a vertex coloring $c(u)$ by defining $c(u) = \sum_{u \sim e} w(e)$ for every vertex $u \in V(G)$. A k-edge-weighting of a graph G is *vertex-coloring* if the induced vertex-coloring is proper, i.e. $c(u) \neq c(v)$, and we say G admits a *vertex-coloring k-edge-weighting*.

A k-edge-weighting can also be viewed as a partition of edges into sets $\{S_1, \cdots, S_k\}$. For each vertex v, let X_v denote the multiset where the elements are the weightings of the edges incident with v, and multiplicity of i in X_v which is the number of edges incident to v in S_i. An edge-weighting is *proper* if no two incident edges receive the same label. An edge-weighting is *adjacent vertex-distinguishing* if for every edge $e = uv$, $X_u \neq X_v$; it is *vertex-distinguishing* if $X_u \neq X_v$ holds for any pair of vertices $u, v \in V(G)$. Proper (adjacent) vertex-distinguishing edge-weighting has been studied by many researchers [4,5,7], and is reminiscent of harmonious colorings [10]. Clearly, if a k-edge-weighting is vertex-coloring, then it is adjacent vertex-distinguishing. However, the converse may not hold.

If a graph has an edge as a component, it cannot have an adjacent vertex-distinguishing or vertex-coloring edge -weighting. Thus in this paper we only consider graphs without an edge component, and refer to them as *nice graphs*. The initial study of vertex-coloring and adjacent vertex-distinguishing edge-weightings posed the following conjecture.

Conjecture 1.1. (Karoński *et al.* [13]) Every nice graph admits a vertex-coloring 3-edge-weighting.

Furthermore, Karoński *et al.* proved that this conjecture holds for 3-colorable graphs (Theorem 1 of [13]). Chang *et al.* [8] considered bipartite graphs $G = (X, Y)$, and proved that if $|X||Y|$ is even the graph admits a vertex-coloring 2-edge-weighting. Lu *et al.* [9] improved this result, by showing that all 3-connected bipartite graphs have vertex-coloring 2-edge-weighting. For general graphs, Addario-Berry *et al.* [2] showed that every nice graph admits a vertex-coloring 30-edge-weighting. Addario-Berry *et al.* [3] then proved that every nice graph permits a vertex-coloring 16-edge-weighting. Wang and Yu [15] improved this bound to 13, and Kalkowski *et al.* [12] proved that every nice graph permits a vertex-coloring 5-edge-weighting.

On the other hand, there are many results for adjacent vertex-distinguishing edge-weighting. Every nice graph permits an adjacent vertex-distinguishing 213-edge-weighting and graphs with minimum degree at least 10^{99} permit an adjacent vertex-distinguishing 30-edge-weighting [13]; and every nice graph permits an adjacent vertex-distinguishing 4-edge-weighting, and that graphs of minimum degree at least 1000 permit an adjacent vertex-distinguishing 3-edge-weighting [1].

There is a close relationship between 2-edge-weighting and a special list factor. If $L : V(G) \mapsto 2^N$ is a set function, a *list factor* (or *L-factor* for short) of a graph G is a spanning subgraph H such that $d_H(v) \in L(v)$ for all $v \in V(G)$.

In general, an L-factor problem is NP-complete, even when G is bipartite. A comprehensive investigation of L-factors was carried out by Lovász [14]. An L-factor is a spanning subgraph with degrees from specified sets. When each $L(v)$ is an interval, L-factors are the same as usual degree factors. For instance, let f and g be nonnegative integer-valued functions on $V(G)$ with $f \geq g$ and $L(v) = [g(v), f(v)]$ for $v \in V(G)$. An L-factor is then exactly