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Implicitly Restarted Refined Generalised Arnoldi Method with Deflation for the Polynomial Eigenvalue Problem

Wei Wei and Hua Dai*

Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China.

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Abstract. Based on the generalised Arnoldi procedure, we develop an implicitly restarted generalised Arnoldi method for solving the large-scale polynomial eigenvalue problem. By combining implicit restarting with the refinement scheme, we present an implicitly restarted refined generalised Arnoldi (IRGAR) method. To avoid repeated converged eigenpairs in the later iteration, we develop a novel non-equivalence low-rank deflation technique and propose a deflated and implicitly restarted refined generalised Arnoldi method (DIRGAR). Some numerical experiments show that this DIRGAR method is efficient and robust.

AMS subject classifications: 65F15

Key words: Polynomial eigenvalue problem, generalised Arnoldi method, refinement, implicit restarting, non-equivalence low-rank deflation.

1. Introduction

We consider the matrix polynomial

$$P(\lambda) = \lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + A_0, \tag{1.1}$$

where the coefficient matrices $A_i (0 \le i \le d)$ are $n \times n$ large and sparse. The scalar λ is said to be an eigenvalue of $P(\lambda)$ if $det(P(\lambda)) = 0$, where $det(P(\lambda))$ denotes the determinant of the matrix $P(\lambda)$. The nonzero vectors x, y are said to be the right and left eigenvectors corresponding to the eigenvalue λ if

$$P(\lambda)x = 0 \quad \text{and} \quad y^H P(\lambda) = 0\,, \tag{1.2}$$

respectively. This is the well known polynomial eigenvalue problem (PEP). For convenience, the two-tuple (λ, x) or the triplet (λ, x, y) is used to denote an eigenpair of the problem (1.2).

*Corresponding author. *Email addresses:* w.wei@nuaa.edu.cn (W. Wei), hdai@nuaa.edu.cn (H. Dai)

The PEP reduces to the generalised eigenvalue problem if $d = 1$; and to the quadratic eigenvalue problem (QEP) if $d = 2$, which is one of the most important cases. Thus for the QEP Tisseur & Meerbergen [31] surveyed many applications, properties and numerical methods. The cubic eigenvalue problem (CEP) when $d = 3$ arises in the numerical simulation of the semiconductor quantum dot model [15, 32]; and higher order polynomial eigenvalue problems arise in stability analysis in control systems [12], the simulation of the three-dimensional pyramid quantum dot heterostructure [16], and in structural dynamic analysis via the dynamic element method [26] or the least squares element method [27].

Gohberg *et al.* [9] established the mathematical theory for matrix polynomials. Chu [6], Dedieu & Tisseur [7], Higham & Tisseur [13] considered perturbation analysis for the PEP. Tisseur [30], and Higham *et al.* [10] and Lawrence & Corless [21] analysed the backward error. Here we consider the computation of some eigenpairs with eigenvalues of largest modulus. If some eigenpairs with eigenvalues nearest to a target σ are desired, one may apply the shift-invert transformation $\tilde{\lambda} = 1/(\lambda - \sigma)$ to the matrix polynomial $P(\lambda)$ and consider the transferred matrix polynomial

$$\tilde{P}(\tilde{\lambda}) = \tilde{\lambda}^d \tilde{A}_d + \tilde{\lambda}^{d-1}\tilde{A}_{d-1} + \cdots + \tilde{\lambda}\tilde{A}_1 + \tilde{A}_0 \,, \tag{1.3}$$

where $\tilde{A}_i = \sum_{j=0}^{i} C_{d-j}^{i-j} \sigma^{i-j} A_{d-j} (0 \le i \le d)$, $C_n^k = \frac{n!}{k!(n-k)!}$. For $\sigma = 0$ (some smallest modulus eigenvalues are desired), such that the new coefficient matrices satisfy $\tilde{A}_i = A_{d-i}$ $(0 \le i \le d)$, one simply inverts the order of the coefficient matrices $A_i (0 \le i \le d)$ in $P(\lambda)$. If the coefficient matrix A_d is singular, then $\lambda = \infty$ is an eigenvalue of the PEP (1.2) — and here we assume that A_d is nonsingular throughout.

The classical approach for solving the PEP (1.2) is to linearise the problem and produce the following generalised eigenvalue problem:

$$Cy = \lambda G y \,, \tag{1.4}$$

where

$$C = \begin{pmatrix} -A_{d-1} & -A_{d-2} & \cdots & -A_0 \\ I & & & \\ & \ddots & & \\ & & I & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} A_d & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix},$$

$$y = \begin{pmatrix} \lambda^{d-1} x \\ \lambda^{d-2} x \\ \vdots \\ x \end{pmatrix}. \tag{1.5}$$

Many different linearisations are possible [1, 9, 11, 23, 24]. Provided A_d is nonsingular, the generalised eigenvalue problem (1.4) may be reduced to the standard eigenvalue problem

$$My = \lambda y, \tag{1.6}$$