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Model Reduction of a Two-Dimensional Kinetic Swarming Model by Operator Projections

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Abstract. Hyperbolic, rotation invariant moment systems are derived for a non-linear kinetic description of two-dimensional Vicsek swarming model. The systems also preserve mass conservation, and numerical experiments show that this approach captures the key features of the model such as shock waves, contact discontinuities, rarefaction waves, vortex formations. If the system order increases, the solutions of the moment systems converge to the solution of the corresponding kinetic equation.

AMS subject classifications: 82C40, 35L60

Key words: Moment method, hyperbolicity, kinetic equation, model reduction, projection operator.

1. Introduction

Swarming is a collective behaviour exhibited by entities, particularly animals of similar size which aggregate together, milling about the same spot, or moving en masse or migrating in some direction — cf. Ref. [1]. Various swarming models have been discussed in literature [8, 11, 17], but numerical methods for such models are not well developed. The first numerical investigation, published only recently by Gamba *et al.* [14], deals with the kinetic description of Vicsek swarming model. On the other hand, kinetic theory, which plays an important role in many applications, has been widely studied — cf. Refs. [9, 10]. The kinetic equation determines the distribution function and, consequently, the transport coefficients. The moment method [5, 16, 24] is a reduction of the kinetic equation based on the expansion of the distribution function into the series of tensorial Hermite polynomials and introduction of balance equations for higher order moments of this function. A major disadvantage of the Grad moment method is the loss of hyperbolicity, which causes the solution blow-up for distributions distant from the equilibrium state — cf. Refs. [7, 25], and

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even the increase of the moment numbers does not help to prevent this effect [6]. Nevertheless, recently various approaches to the Grad moment method for the kinetic equation have been proposed — e.g. numerically regularized moment model of arbitrary order for Boltzmann-BGK equation [5] and high Mach number flow [6], the regularization of one- and multi-dimensional Grad moment systems leading to global hyperbolicity [2–4], quadrature based projection methods for Boltzmann equation [13, 18, 21], similar to 1-D regularization in Ref. [2]. These works contributed to a better understanding of the hyperbolicity of the Grad moment systems. On the other hand, Fan *et al.* [13] developed a general model reduction technique based on operator projections, where time and space derivatives are synchronously projected onto a finite-dimensional weighted polynomial space. Such a method was successfully applied to 1- and 3-D special relativistic Boltzmann equations and the globally hyperbolic moment models of arbitrary order were derived [22, 23].

In this paper we apply the projection operator model reduction method to the two-dimensional kinetic description of the Vicsek swarming model determining a globally hyperbolic moment system of arbitrary order. The moment system is obtained by the Grad type expansion near the equilibrium, and if the ratio ε of micro to macro variables is small, the system with small moment numbers delivers a good approximation for the original kinetic model. The main difficulties here are to find weight functions which would determine appropriate function spaces, suitable bases and projection operators in these spaces.

The paper is organized as follows. Section 2 introduces the kinetic and macroscopic equations for Vicsek model. Section 3 deals with orthogonal functions and presents globally hyperbolic moment systems of arbitrary order for the kinetic description of Vicsek model. The properties of the moment systems such as hyperbolicity, rotational invariance and mass-conservation are considered in Section 4. Numerical experiments presented in Section 5 confirm the convergence of the hyperbolic moment systems, and Section 6 contains our concluding remarks.

2. Kinetic and Macroscopic Equations for Vicsek Model

2.1. Kinetic equation

Let t denote time, $\boldsymbol{x} \in \mathbb{R}^d$ spatial variable, $\boldsymbol{\omega}$ unit velocity vector, σ scaled diffusion constant describing the intensity of noise in Brownian motion, $f = f(t, \boldsymbol{x}, \boldsymbol{\omega})$ the particle distribution function, and let $\boldsymbol{J}(t, \boldsymbol{x})$ refer to the mean flux at $\boldsymbol{x}$ — i.e.

$$\boldsymbol{J}(t,\boldsymbol{x}) := \int_{\mathbb{R}^d} \int_{S^{d-1}} K(\boldsymbol{y}-\boldsymbol{x})\boldsymbol{\omega} f(t,\boldsymbol{y},\boldsymbol{\omega}) \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{\omega}, \tag{2.1}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$, $K(\boldsymbol{x}) = \mathbf{1}_{|\boldsymbol{x}|<R}$ the characteristic function of the ball $B(0,R) = \{\boldsymbol{x} : |\boldsymbol{x}| \leq R\}$, with R being the radius of alignment interaction between particles. By $\boldsymbol{F}[f](t, \boldsymbol{x}, \boldsymbol{\omega})$ we denote the mean-field interaction force between the particles,

$$\boldsymbol{F}[f](t,\boldsymbol{x},\boldsymbol{\omega}) := (\mathrm{Id} - \boldsymbol{\omega} \otimes \boldsymbol{\omega})\Omega(t,\boldsymbol{x}),$$