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Application of Reproducing Kernel Hilbert Spaces to a Minimization Problem with Prescribed Nodes

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Abstract. The theory of reproducing kernel Hilbert spaces is applied to a minimization problem with prescribed nodes. We re-prove and generalize some results previously obtained by Gunawan *et al.* [2,3], and also discuss the Hölder continuity of the solution to the problem.

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1. Introduction

Consider a Hilbert space H_α $(0 \le \alpha < \infty)$ defined by the set of functions f on $[0,1]^d$ of form

$$f(x_1,\dots,x_n) := \sum_{m_1,\dots,m_d \in \mathbb{N}} a_{m_1\cdots m_d} \sin(m_1\pi x_1)\cdots\sin(m_d\pi x_d)\,,$$

for which

$$\|f\|_{H_\alpha} := \frac{\pi^{2\alpha}}{2^d} \sum_{m_1,\dots,m_d \in \mathbb{N}} (m_1^2+\cdots+m_d^2)^\alpha |a_{m_1\cdots m_d}|^2 < \infty\,.$$

The above norm is induced from the inner product

$$\langle f,g\rangle_{H_\alpha} = \frac{\pi^{2\alpha}}{2^d} \sum_{m_1,\dots,m_d \in \mathbb{N}} (m_1^2+\cdots+m_d^2)^\alpha a_{m_1\cdots m_d} b_{m_1\cdots m_d}\,,$$

where $a_{m_1\cdots m_d}$ and $b_{m_1\cdots m_d}$ are the coefficients of f and g, respectively. We now proceed to solve the following minimization problem on $\mathbb{R}^d$:

$$\text{Minimize } \|f\|_{H_\alpha}$$

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subject to the prescribed nodes

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \dots, N ,$$

where $\mathbf{p}_k := (p_{k1}, \dots, p_{kd}) \in (0,1)^d$ and $c_k \in \mathbb{R}$ are given. (Note that the points $\mathbf{p}_k$'s are inside the unit cube $[0,1]^d$.) The 1- and 2-dimensional cases have been studied by Gunawan *et al.* [2,3], who show inter alia that the value $\alpha > d/2$ is a necessary and sufficient condition for the solution to be continuous. As one might expect, the larger the value of α, the smoother the solution. Here we use the theory of reproducing kernel Hilbert spaces to study the problem in a more general setting.

Our first result is the following theorem.

Theorem 1.1. *Let $\alpha > d/2$. The solution to the minimization problem*

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(p_1, \dots, p_d) = 1 ,$$

is given by

$$F(x_1, \dots, x_d) := A \sum_{m_1,\dots,m_d \in \mathbb{N}} \frac{\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d)}{(m_1^2 + \cdots + m_d^2)^\alpha} \sin(m_1 \pi x_1) \cdots \sin(m_d \pi x_d) ,$$

where $$A^{-1} := \sum_{m_1,\dots,m_d \in \mathbb{N}} (\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d))^2 / (m_1^2 + \cdots + m_d^2)^\alpha .$$

The proof of this theorem is given in Section 2, where a more general result is also presented. In Section 3, we consider the Hölder continuity of the solution, by using the relationship between Besov and modulation spaces.

2. Main Results

Let E be a compact subspace of $\mathbb{R}^d$ containing at least N points, and $K : E \times E \to \mathbb{F}$ a positive definite kernel where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let H_K denote the corresponding reproducing kernel Hilbert space, which is defined as the completion of the pre-Hilbert space $H_K^0 := \operatorname{span}_{\mathbb{F}}\{K(\cdot, \mathbf{p}) : \mathbf{p} \in E\}$, equipped uniquely with the inner product $\langle \cdot, \cdot \rangle_{H_K^0}$ so that $\forall\, \mathbf{p}, \mathbf{q} \in E$

$$\langle K(\cdot, \mathbf{p}), K(\cdot, \mathbf{q}) \rangle_{H_K^0} = K(\mathbf{q}, \mathbf{p}) .$$

A well-known fact in the theory of reproducing kernel Hilbert spaces is that

$$f(\mathbf{p}) = \langle f, K(\cdot, \mathbf{p}) \rangle_{H_K}$$

for every $f \in H_K$ and $\mathbf{p} \in E$. Accordingly, we have the following proposition.