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A Fast Semi-Implicit Level Set Method for Curvature Dependent Flows with an Application to Limit Cycles Extraction in Dynamical Systems

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Abstract. We propose a new semi-implicit level set approach to a class of curvature dependent flows. The method generalizes a recent algorithm proposed for the motion by mean curvature where the interface is updated by solving the Rudin-Osher-Fatemi (ROF) model for image regularization. Our proposal is general enough so that one can easily extend and apply the method to other curvature dependent motions. Since the derivation is based on a semi-implicit time discretization, this suggests that the numerical scheme is stable even using a time-step significantly larger than that of the corresponding explicit method. As an interesting application of the numerical approach, we propose a new variational approach for extracting limit cycles in dynamical systems. The resulting algorithm can automatically detect multiple limit cycles staying inside the initial guess with no condition imposed on the number nor the location of the limit cycles. Further, we also propose in this work an Eulerian approach based on the level set method to test if the limit cycles are stable or unstable.

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1 Introduction

Curvature dependent flows are interesting not only mathematically but also computationally. Numerically, the motion of a parametrized curve can be determined by solving the corresponding ordinary differential equation (ODE) for each discretized point

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on the curve. However, as the curve evolves, such explicit representation might require re-meshing to obtain a better interface sampling. And worst, it might also need careful numerical surgery if there is a topological change when the curve splits into multiple disjoint components. Another class of numerical algorithms is implicit methods based on the level set approach, including the approach in [44] by regularizing the curvature term using the Laplacian of the level set function, a modified MBO approach [12] which generates appropriate motion by diffusion, and the variational approach in [7,8] by applying the ROF functional [40] from the image processing community.

In this paper, we propose a new semi-implicit scheme for computing a class of curvature dependent evolutions of a codimension-one surface in the level set formulation. We consider the evolution of the level set equation

$$\phi_t = |\nabla\phi| v_n(\kappa) = |\nabla\phi| v_n \left[\nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right], \tag{1.1}$$

where the normal velocity $v_n(\kappa)$ defined on the interface satisfies $v_n(\kappa)\kappa \geq 0$. Mathematically, this constraint on the normal velocity gives a stability condition in the curve evolution. To see this, one can show that (for example in [41]) if $\alpha(s)$ denotes the arclength of a curve $\gamma(s;t) = \{(x(s;t),y(s;t)) : t \geq 0\}$ parametrized by the parameter s at a given time t, then we have $d\alpha = g(s;t)ds$ where $g(s;t) = \sqrt{x_s^2 + y_s^2}$ and

$$g_t(s;t) = -g(s;t) v_n(\kappa)\kappa.$$

Therefore, the condition $v_n(\kappa)\kappa \geq 0$ actually imposes a condition that the curve should collapse under its evolution in time.

Our approach is developed based on a regularization technique. However, unlike the regularization by a standard Laplacian as in the approach in [44], we propose to regularize the geometrical flow by adding and subtracting a *curvature* term. Applying the algorithm to the motion by mean curvature, we will show that our scheme reduces to the variational functional in [7,8]. From this point of view, our method can be regarded as a generalization of [7,8]. However, unlike these approaches, our interpretation allows us to easily extend the algorithm to deal with a much wider class of curvature dependent flows. Moreover, since our approach is developed based on a semi-implicit time-discretization, the resulting numerical method has a relatively large time-step size. Even though we do not have any theoretical estimate on the required stability condition since the regularization is in fact a nonlinear one, we find that the numerical solution gives a stable evolution of the interface with a marching step significantly larger than the one by a corresponding explicit scheme. This interesting property has not yet been reported in the work of [7,8] which is solely based on the property of the ROF functional.

As an interesting and important application of the proposed algorithm, we develop and apply a variational method for extracting invariant manifolds in dynamical systems. One kind of important invariant manifolds is the Lagrangian coherent structures (LCS). The main idea in determining the LCS is to partition the space-time domain into different