Commun. Comput. Phys.
doi: 10.4208/cicp.120314.230115a

Vol. **18**, No. 2, pp. 469-488
August 2015

Preconditioned Iterative Methods for Two-Dimensional Space-Fractional Diffusion Equations

Xiao-Qing Jin[1], Fu-Rong Lin[2] and Zhi Zhao[1,*]

[1] *Department of Mathematics, University of Macau, Macao 999078, China.*
[2] *Department of Mathematics, Shantou University, Shantou 515063, China.*

Received 12 March 2014; Accepted (in revised version) 23 January 2015

Abstract. In this paper, preconditioned iterative methods for solving two-dimensional space-fractional diffusion equations are considered. The fractional diffusion equation is discretized by a second-order finite difference scheme, namely, the Crank-Nicolson weighted and shifted Grünwald difference (CN-WSGD) scheme proposed in [W. Tian, H. Zhou and W. Deng, *A class of second order difference approximation for solving space fractional diffusion equations*, Math. Comp., 84 (2015) 1703-1727]. For the discretized linear systems, we first propose preconditioned iterative methods to solve them. Then we apply the D'Yakonov ADI scheme to split the linear systems and solve the obtained splitting systems by iterative methods. Two preconditioned iterative methods, the preconditioned generalized minimal residual (preconditioned GMRES) method and the preconditioned conjugate gradient normal residual (preconditioned CGNR) method, are proposed to solve relevant linear systems. By fully exploiting the structure of the coefficient matrix, we design two special kinds of preconditioners, which are easily constructed and are able to accelerate convergence of iterative solvers. Numerical results show the efficiency of our preconditioners.

AMS subject classifications: 65F10, 65L06, 65U05

Key words: Fractional diffusion equation, CN-WSGD scheme, preconditioned GMRES method, preconditioned CGNR method, Toeplitz matrix, fast Fourier transform.

1 Introduction

In this paper, we consider the following initial boundary value problem of two-dimensional space-fractional diffusion equation [25,43]:

*Corresponding author. *Email addresses:* `xqjin@umac.mo` (X.-Q. Jin), `frlin@stu.edu.cn` (F.-R. Lin), `zhaozhi231@163.com` (Z. Zhao)

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = d_+(x,y,t)\,{}_aD_x^\alpha u(x,y,t) + d_-(x,y,t)\,{}_xD_b^\alpha u(x,y,t) \\ \qquad\qquad + e_+(x,y,t)\,{}_cD_y^\beta u(x,y,t) + e_-(x,y,t)\,{}_yD_d^\beta u(x,y,t) + f(x,y,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,y,t)\in\Omega\times(0,T], \\ u(x,y,0) = u_0(x,y), \qquad (x,y)\in\overline{\Omega}, \\ u(x,y,t) = 0, \qquad (x,y,t)\in\partial\Omega\times[0,T], \end{cases} \tag{1.1}$$

where $1<\alpha,\beta<2$, $d_\pm(x,y,t), e_\pm(x,y,t)\geq 0$, and $\Omega=(a,b)\times(c,d)$. In addition, ${}_aD_x^\alpha$, ${}_xD_b^\alpha$, ${}_cD_y^\beta$, and ${}_yD_d^\beta$ are the left-sided and right-sided Riemann-Liouville fractional derivatives defined by

$$\begin{cases} {}_aD_x^\alpha u(x,y,t) = \dfrac{1}{\Gamma(2-\alpha)}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\displaystyle\int_a^x \dfrac{u(\xi,y,t)}{(x-\xi)^{\alpha-1}}\mathrm{d}\xi, \\ {}_xD_b^\alpha u(x,y,t) = \dfrac{1}{\Gamma(2-\alpha)}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\displaystyle\int_x^b \dfrac{u(\xi,y,t)}{(\xi-x)^{\alpha-1}}\mathrm{d}\xi, \\ {}_cD_y^\beta u(x,y,t) = \dfrac{1}{\Gamma(2-\beta)}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\displaystyle\int_c^y \dfrac{u(x,\eta,t)}{(y-\eta)^{\beta-1}}\mathrm{d}\eta, \\ {}_yD_d^\beta u(x,y,t) = \dfrac{1}{\Gamma(2-\beta)}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\displaystyle\int_y^d \dfrac{u(x,\eta,t)}{(\eta-y)^{\beta-1}}\mathrm{d}\eta. \end{cases}$$

Fractional diffusion equations (FDEs) have many applications in various fields, such as fluid mechanics, finance, biology, turbulent flow, electrochemistry physics, and image processing [2, 8, 18, 22, 28, 31, 33, 35, 36]. Since closed-form solutions are only available for very few FDEs, a variety of numerical methods for FDEs have been developed in the last decade [12–16, 21, 24–27, 32, 37, 38, 41, 44].

In 2004 and 2006, Meerschaert and Tadjeran proposed a shifted Grünwald discretization scheme to approximate the FDE with a left-sided fractional derivative and the FDE with two-sided fractional derivatives, respectively [24,25]. In 2006, Tadjeran, Meerschaert, and Scheffler proposed a second-order accurate numerical scheme based on the classical Crank-Nicolson method and the Richardson extrapolation scheme [38]. Their methods were proved to be unconditionally stable. In 2012, Tian, Zhou, and Deng proposed a weighted and shifted Grünwald difference (WSGD) scheme to approximate left-sided and right-sided Riemann-Liouville fractional derivatives, and obtained a class of second-order accurate CN-WSGD schemes [39].

Although the discretization of second-order partial differential equations leads to linear systems with sparse coefficient matrices, the matrices corresponding to FDEs are full. Fortunately, the coefficient matrices of the above mentioned methods possess a special Toeplitz-like structure: it can be written as a sum of diagonal-multiply-Toeplitz matrices [39, 41]. Thus the storage requirement is significantly reduced from $\mathcal{O}(N^2)$ to