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The Pullback Asymptotic Behavior of the Solutions for 2D Nonautonomous *G*-Navier-Stokes Equations

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Abstract. The pullback asymptotic behavior of the solutions for 2D Nonautonomous G-Navier-Stokes equations is studied, and the existence of its L^2-pullback attractors on some bounded domains with Dirichlet boundary conditions is investigated by using the measure of noncompactness. Then the estimation of the fractal dimensions for the 2D G-Navier-Stokes equations is given.

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Key words: Pullback attractor, G-Navier-Stokes equation, fractal dimension, the measure of noncompactness, bounded domains.

1 Introduction

The Navier-Stokes equations have received much attention over past decades due to their importance in the understanding of fluids motion and turbulence. In this paper, we consider the 2D nonautonomous G-Navier-Stokes equations on some bounded domain $\Omega\subset\mathbf{R}^2$ with Dirichlet boundary conditions, which has the following form, (see Roh [1,2] and Jiang and Hou [3])

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t), \qquad \text{in } \Omega\times(0,\infty), \tag{1.1a}$$

$$\nabla\cdot(gu) = 0, \qquad \text{in } \Omega\times(0,\infty), \tag{1.1b}$$

$$u(x,t) = 0, \qquad \text{on } \partial\Omega, \tag{1.1c}$$

$$u(x,0) = u_0(x), \qquad \text{in } \Omega, \tag{1.1d}$$

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where $u(x,t)\in\mathbf{R}^2$ and $p(x,t)\in\mathbf{R}$ denote the velocity and the pressure, $\nu>0$ and $f=f(x,t)\in(L^2(\Omega))^2$ is the time-dependent external force. $0<m_0\le g=g(x_1,x_2)\le M_0$. Here, $g=g(x_1,x_2)$ is a suitable real-valued smooth function. When $g=1$, the Eqs. (1.1) become the usual 2D Navier-Stokes equations. In [12], Raugel and Sell proved global existence of strong solutions for large initial data and forcing terms in thin three dimensional domains. In 2005, Roh applied Raugel and Sell methods on $\Omega_g=\Omega_2\times(0,g)$ and derive the 2D G-Navier-Stokes equations form 3D Navier-Stokes equation in [1,2]. In this paper, our aim is to study the long-time behaviour of weak solutions of problem (1.1) by using the theory of pullback attractors. This theory is a natural generalization of the theory of global attractors developed to study autonomous dynamical systems (see [3–12]), and the theory of pullback attractors has an advantage over the theory of uniform attractors (see [13]) allowing the nonautonomous term to be an arbitrary in suitable norms.

Recently, Caraballo in [14] introduces the notion of pullback $\mathcal{D}$-attractor for nonautonomous dynamical systems and prove the existence of pullback $\mathcal{D}$-attractor on some unbounded domains by using the energy equation method. Langa in [15] obtains fractal dimension for 2D N-S equation. Motivated by some ideas in [14,15], we present a new equivalent condition (PC) for pullback $\mathcal{D}$-asymptotically compact by using the measure of noncompactness. In this paper, we prove the existence of pullback attractor and estimate its fractal dimension for 2D G-N-S equation on some bounded domains.

This paper is organized as follows: in Section 1, we recall some basic notations and results for 2D G-Navier-Stokes equations and the concept about the measure of noncompactness. In Section 2, we apply the theory of the measure of noncompactness to obtain the existence of the pullback attractor for non-autonomous G-N-S equation on some bounded domains; then, In Section 3, we estimate the fractal dimension of pullback attractor for 2D G-N-S equation on some bounded domains.

2 Preliminaries

Now, we assume that the Poincaré inequality holds on Ω, there exists an $\lambda_1>0$ such that

$$\int_\Omega \phi^2 g dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 g dx, \quad \forall \phi\in H_0^1(\Omega). \tag{2.1}$$

The mathematical frameworks of (1.1) is the following:

- Let $L^2(g)=(L^2(\Omega))^2$ with the inner products,

$$(u,v)=\int_\Omega u\cdot v g dx \quad \text{and norms} \quad |\cdot|=(\cdot,\cdot)^{\frac{1}{2}}, \quad u,v\in L^2(g).$$

- Let $H_0^1(g)=(H_0^1(\Omega))^2$, which is endowed with the inner products,

$$((u,v))=\int_\Omega \sum_{j=1}^2 \nabla u_j\cdot\nabla v_j g dx,$$