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An Alternative Lopsided PMHSS Iteration Method for Complex Symmetric Systems of Linear Equations

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Abstract. An iterative method for complex symmetric systems of linear equations is proposed. Estimates for the spectral radius of the method are obtained and sufficient convergence conditions are established. Numerical experiments show the efficiency of the method for linear systems with a dominant real part.

AMS subject classifications: 65F10, 65F50

Key words: Complex symmetric linear system, iterative method.

1. Introduction

We consider the system of linear equations

$$Ax = b, \quad x, b \in \mathbb{C}^n, \tag{1.1}$$

with a matrix $A = W + iT$, $i^2 = -1$, where $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ are, respectively, symmetric positive definite and symmetric positive semi-definite matrices. Systems of linear equations with complex symmetric matrices often arise in scientific and engineering applications [1, 3, 13]. A lot of efforts have been spent on the development of approximation methods for their solution — cf. Refs. [11,12,15,17,18,22,26]. Nevertheless, in spite of the fact that the matrix $A \in \mathbb{C}^{n\times n}$ has a natural Hermitian and skew-Hermitian splitting (HSS),

$$A = H + S,$$

with

$$H = \frac{1}{2}(A + A^*) = W, \quad S = \frac{1}{2}(A - A^*) = iT,$$

the HSS iteration method [7] for system (1.1) experiences certain difficulties — cf. Refs. [2, 6–10, 27]. To overcome these problems, Bai *et al.* [3–5] developed a modified HSS method

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(MHSS) and a preconditioned modified HSS method (PMHSS) (see also [16]). The PMHSS iteration method works as follows. Choose a positive constant α, a symmetric positive definite matrix $V \in \mathbb{R}^{n\times n}$, an initial approximation $x^{(0)} \in \mathbb{C}^n$ and construct the sequence $\{x^{(k)}\}$ by

$$\begin{aligned}
&(\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - iT)x^{(k)} + b,\\
&(\alpha V + T)x^{(k+1)} = (\alpha V + i\, W)x^{(k+\frac{1}{2})} - ib,\\
&k = 0, 1, \cdots,
\end{aligned}$$

stopping the procedure at some point. In particular, if $V = I$ then PMHSS becomes the MHSS iteration method. Moreover, for any $\alpha > 0$ and for any $x^{(0)}$, the PMHSS iteration method converges to the unique solution of the system (1.1) [4]. Starting with the PMHSS and the lopsided HSS iteration method (LHSS) [20,25], Li *et al.* [21] proposed the following lopsided PMHSS iteration method (LPMHSS):

$$\begin{aligned}
&Wx^{(k+\frac{1}{2})} = -i\, Tx^{(k)} + b,\\
&(\alpha V + T)x^{(k+1)} = (\alpha V + i\, W)x^{(k+\frac{1}{2})} - ib,\\
&k = 0, 1, \cdots .
\end{aligned}$$

The theoretical analysis shows that the LPMHSS method converges to the unique solution of (1.1) under weak restrictions on α. In addition, an upper bound for the spectral radius of the LPMHSS iteration matrix and the quasi-optimal parameter α^*, minimising the upper bound have been derived [21]. On the other hand, in [23] the non-Hermitian and non-normal positive definite systems have been treated by another lopsided HSS iteration method:

$$\begin{aligned}
&Hx^{(k+\frac{1}{2})} = -Sx^{(k)} + b,\\
&(\alpha I + H)x^{(k+1)} = (\alpha I - S)x^{(k+\frac{1}{2})} + b,\\
&k = 0, 1, \cdots .
\end{aligned} \tag{1.2}$$

For any $\alpha > 0$ and any $x^{(0)}$, this method converges to the unique solution of the system (1.1) [19, 23], and if the coefficient matrix A has a dominant Hermitian part — i.e. if $\|H\| \gg \|S\|$, then the method (1.2) converges faster than the HSS method.

In this paper we study an alternative lopsided preconditioned modified Hermitian and skew-Hermitian splitting (ALPMHSS) iteration method — viz.

$$\begin{aligned}
&Wx^{(k+\frac{1}{2})} = -i\, Tx^{(k)} + b,\\
&(\alpha V + W)x^{(k+1)} = (\alpha V - iT)x^{(k+\frac{1}{2})} + b,\\
&k = 0, 1, \cdots ,
\end{aligned}$$

where constant α and matrix V are as above.

Since matrices W and $\alpha V + W$ are symmetric positive definite, one can use various methods to solve two linear sub-systems, including Cholesky's factorization and preconditioned