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A NUMERICAL METHOD FOR SOLVING THE VARIABLE COEFFICIENT WAVE EQUATION WITH INTERFACE JUMP CONDITIONS

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Abstract. Wave equations with interface jump conditions have wide applications in engineering and science, for example in acoustics, elastodynamics, seismology, and electromagnetics. In this paper, an efficient non-traditional finite element method with non-body-fitted grids is proposed to solve variable coefficient wave equations with interface jump conditions. Numerical experiments show that this method is approximately second order accurate both in the L^∞ norm and L^2 norm for piecewise smooth solutions.

Key words. Non-traditional finite element method, wave equation, jump condition, variable coefficient.

1. Introduction

Problems involving wave equations with interfaces have a wide variety of applications in science and engineering, for example in acoustics, seismology and electromagnetics. Designing highly effective and computational efficient methods for these problems is nontrivial.

Before studying the wave interface problems, one needs to study the method for solving the elliptic interface problem since that is one of the major challenges in the problem. Therefore we first summarize the past work on elliptic interface problems below.

For nearly four decades, extensive research has been performed in the area of numerical solutions of elliptic equations with discontinuous coefficients and singular sources on Cartesian grids. The choice of uniform Cartesian grids saves the cost of mesh generation. It started with the pioneering work of Peskin [1] on the first order accurate immersed boundary method developed to simulate the pattern of blood flow in the heart.

Also, a great amount of work has been done to use finite difference methods on elliptic interface problems. The main idea is to use difference schemes and stencils near the interface to incorporate the jump conditions and interface in the Taylor expansions. Using finite difference schemes requires the use of high order derivatives of jump conditions and interface conditions. LeVeque and Li proposed the immersed interface method for solving elliptic equations with discontinuous coefficients and singular sources [2]. This method incorporates the interface conditions in both solution and flux, $[u] \neq 0$ and $[\beta u_n] \neq 0$, into the finite difference stencil resulting in second order accuracy. The method produces a linear system that is sparse, but may not be symmetric or positive definite if there is a jump in the coefficient. Detailed information about the IIM can be found in [3].

In [4], the matched interface and boundary method (MIB) was proposed to solve elliptic equations with smooth interfaces. In [5], the MIB method was generalized

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for problems involving sharp-edged interfaces. In [6], the MIB method was generalized for problems involving triple junction points. This method has achieved 2nd order accuracy in the L^∞ norm even for sharp-edged interfaces.

In [7] and [8], the immersed finite element method (IFEM) was developed using non-body fitted Cartesian meshes for homogeneous jump conditions. The method was extended to treat non-homogeneous jump conditions in [9]. The partially penalized IFEM was developed in [10].

Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [11, 12] for two dimensional problems and [13] for three dimensional problems. Furthermore, Discontinuous Galerkin method [14] can be used to solve elliptic interface problems. Both the mesh and polynomial degree can be adaptively refined in a remeshing scheme [15]. Recently, the gradient recovery method [30–32] was introduced for accurate gradient computation.

Some theoretical discussions about interface problems can be found in [16] and [17].

This paper is based on the Petrov-Galerkin type non-traditional finite element method for solving elliptic interface problems that was first introduced in [18] and improved in [19] and [20]. [19] extended the original method to include the case of sharp edged interfaces with matrix coefficients. This extension improved the accuracy for sharp edged interfaces from 0.8th order to nearly second order. In [18] and [19], if the interface hits a grid point exactly, it is perturbed away. [20] treats this case carefully without perturbation. The second improvement in [20] is that not only Dirichlet but also Neumann boundary conditions are considered. The third improvement is that the coefficient matrix data can only be given at the grid points, not as an analytic function. In [21], the method was extended to three dimensions. The extension for the elasticity equations can be found in [22]. The method for multi-domain problems can be found in [23]. Some other extensions can be found in [24] and [25]. The method has two advantages: first, this method uses non-body-fitted grid so that the cost of mesh generation can be saved. Second, compared with other methods the method is easier to implement for complicated problems with non-homogeneous jump conditions and matrix coefficients.

In [26], the second order accurate immersed interface method is used to solve the wave equation with interface jump conditions. The wave equation is rewritten as a first order system. In [27], the correction function method is proposed to solve the wave equation with interface jump conditions. The result is excellent with 4th order accuracy. However, the scope of the work is for constant coefficient case only. On both sides of the interface, it is the Laplacian operator. The scope of our work in this paper is the variable matrix coefficient case, which has wider applications.

2. Formulation and Numerical Method

2.1. Problem Definition and Weak Formulation. In this paper, we solve the wave equation with discontinuous variable matrix coefficients along the interface. Consider a rectangular domain $\Omega = (x_{min}, x_{max}) \times (y_{min}, y_{max})$. Γ is an interface prescribed by the zero level-set $\{(x, y) \in \Omega \mid \phi(x, y) = 0\}$ of a level-set function $\phi(x, y)$. The unit normal vector of Γ is $\boldsymbol{n} = \frac{\nabla\phi}{|\nabla\phi|}$ pointing from $\Omega^- = \{(x, y) \in \Omega \mid \phi(x, y) < 0\}$ to $\Omega^+ = \{(x, y) \in \Omega \mid \phi(x, y) > 0\}$, see Figure 1. Now the governing equation reads

$$(1) \quad \frac{\partial^2 u(x,y,t)}{\partial t^2} - \bigtriangledown \cdot (\beta(x,y,t) \bigtriangledown u(x,y,t)) \quad = \quad f(x,y,t), \text{ in } (\Omega \setminus \Gamma) \times [0,T],$$