INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 4, Pages 647–667

ORTHOGONAL SPLINE COLLOCATION FOR SINGULARLY PERTURBED REACTION DIFFUSION PROBLEMS IN ONE DIMENSION

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Abstract. An orthogonal spline collocation method (OSCM) with C^1 splines of degree $r \geq 3$ is analyzed for the numerical solution of singularly perturbed reaction diffusion problems in one dimension. The method is applied on a Shishkin mesh and quasi-optimal error estimates in weighted H^m norms for $m = 1, 2$ and in a discrete L^2-norm are derived. These estimates are valid uniformly with respect to the perturbation parameter. The results of numerical experiments are presented for C^1 cubic splines ($r = 3$) and C^1 quintic splines ($r = 5$) to demonstrate the efficacy of the OSCM and confirm our theoretical findings. Further, quasi-optimal *a priori* estimates in L^2, L^∞ and $W^{1,\infty}$-norms are observed in numerical computations. Finally, superconvergence of order $2r - 2$ at the mesh points is observed in the approximate solution and also in its first derivative when $r = 5$.

Key words. Singularly perturbed reaction diffusion problems, orthogonal spline collocation, Shishkin mesh, quasi-optimal global error estimates, superconvergence.

1. Introduction

In this paper, we consider singularly perturbed reaction diffusion problems of the form

$$Lu(x) := -\varepsilon u''(x) + a(x)u(x) = f(x), \quad x \in I \equiv (0, 1), \tag{1}$$

subject to the Dirichlet boundary conditions,

$$u(0) = 0, \quad u(1) = 0, \tag{2}$$

where the parameter ε is such that $0 < \varepsilon \ll 1$. It is assumed that the prescribed functions a and f are smooth on I with

$$a(x) \geq \alpha > 0, \quad x \in \overline{I}, \tag{3}$$

where α is a constant. Problems of this type are ubiquitous in the mathematical modeling of numerous real life phenomena; see, for example, [18, 21] and references therein. The solution of (1)–(2) exhibits a multi scale character. Specifically, there is a thin transition layer, often called a boundary layer, at one or both ends of the interval I, in which the solution varies rapidly, while away from the layer, the solution behaves smoothly and varies gradually. This singular behavior of the solution presents some challenges in the development of methods to solve this type of problem accurately and efficiently. One of the most successful approaches involves the use of layer-adapted meshes such as Shishkin meshes [24], which yield methods that converge uniformly, regardless of the magnitude of the parameter ε; see, for example, [17].

In the literature, much attention has been devoted to the development of numerical methods for the solution of singularly perturbed differential equations which converge uniformly with respect to the parameter ε. This work is chronicled in numerous texts, research papers, such as [2, 17, 15, 18, 20, 21, 22, 23, 25, 27] and

Received by the editors on November 7, 2017.
2000 *Mathematics Subject Classification.* 65L11, 65L60, 65L70.

survey articles [11, 12, 13]. Broadly speaking, there are three principal approaches to solving (1)–(2) numerically, namely; finite difference methods, finite element methods and spline collocation methods. Particular attention has been devoted to the latter class of methods, especially, spline collocation based on smoothest splines such as classical C^2 cubic splines; see, [9, 10]. Invariably, these methods yield approximations of suboptimal accuracy. For example, cubic spline collocation applied to (1)–(2) cannot be more than second order accurate, a fact proved in [3]. A collocation method of optimal accuracy that has been used to solve a variety of problems involving ordinary and partial differential equations is the orthogonal spline collocation method (OCSM); see, for example, [1]. The popularity of the orthogonal spline collocation approach is due to its conceptual simplicity, wide range of applicability and ease of implementation. This method, often called spline collocation at Gauss points in the literature, was originally formulated and analyzed over 40 years ago in the seminal paper [4], but has seen little use in the numerical solution of singularly perturbed problems.

In 1980, Flaherty and Mathon [8] used collocation methods with either piecewise polynomials or splines in tension to obtain accurate approximations of one dimensional singularly perturbed boundary value problems. In their paper, the authors state "*Unfortunately, collocation at the Gauss-Legendre points with piecewise polynomials is known to behave rather poorly on singularly-perturbed problems for any partition, where ε is much smaller than the minimum subinterval length.*" In an attempt to generalize the results of [4] to (1)-(2), the authors of [16] considered collocation methods with C^1-quadratic splines using a modified Shishkin mesh and derived almost second order convergence in the maximum norm. In their paper, the authors state, "*C^1-splines with arbitrary order for such problems presents an open task*". In the present article, we develop and analyze an OSCM with C^1 splines of order $r \geq 3$ for solving (1)-(2) using Shishkin meshes in weaker norms. In particular, we derive quasi-optimal estimates of order $(N^{-1}\log N)^{r+1-m}$ in weighted H^m-norms, $m = 1, 2$, and of order $(N^{-1}\log N)^{r+1}$ in a discrete L^2 norm, where N is the number of mesh subintervals. These estimates are valid uniformly with respect to the perturbation parameter ε. The results of numerical experiments confirm our theoretical findings. It is also observed from numerical experiments that error estimates in L^2, L^∞ and $W^{1,\infty}$-norms are quasi-optimal, which are uniform in ε. Moreover, they demonstrate the exceptional performance of the OSCM, especially its ability to yield high order approximations systematically. In [26], the OSCM with $r = 3$ on a Shishkin mesh is considered but, using a different analysis, it is proved that the convergence rate in $L^\infty(I)$ is $O(N^{-4}\log^5 N)$.

A brief outline of this paper is as follows. In Section 2, we introduce notation and some basic results used in the convergence analysis. In Section 3, the OSCM employing a Shishkin mesh is applied to discretize (1)–(2). Section 4 is devoted to the convergence analysis. In Section 5, the results of numerical experiments are presented which illustrate the efficacy of the method and confirm the analytical results. Moreover, they exhibit superconvergence properties of the method which, while anticipated, are as yet unproven. Concluding remarks are given in Section 6.

2. Preliminaries

2.1. Notation. Let $\pi_h = \{x_j\}_{j=0}^N$ denote a partition of $\overline{I} = [0, 1]$ with

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1.$$