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Non-Existence of Standard Wave Operators for Fractional Laplacian and Slowly Decaying Potentials

Atsuhide Ishida*

Department of Liberal Arts, Faculty of Engineering, Tokyo University of Science, 3-1 Niijuku, 6-chome, Katsushika-ku, Tokyo 125-8585, Japan.

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Abstract. Quantum systems described by the fractional powers of the negative Laplacian and the interaction potentials are considered. If the potential function slowly decays and the Dollard-type modified wave operators exist and are asymptotically complete, we prove that the factional Laplacian does not possess the standard wave operators. This result suggests the borderline between the short- and long-range behaviour.

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1. Introduction

We study the scattering phenomena for the fractional powers of the negative Laplacian. Let $D=-i\nabla=-i(\partial_{x_1},\cdots,\partial_{x_n})$ be the momentum operator. For $1/2<\rho\leqslant 1$, the fractional negative Laplacian $\omega_\rho(D)$ on the space $L^2(\mathbb{R}^n)$ is defined by the Fourier multiplier with the symbol

$$\omega_\rho(\xi)=|\xi|^{2\rho}/(2\rho).$$

More exactly, let ϕ belong to the Sobolev space $\mathscr{D}(\omega_\rho(D))=H^{2\rho}(\mathbb{R}^n)$ of order 2ρ and let $\mathscr{F}$ denote the Fourier transform on $\mathbb{R}^n$.

The operator $\omega_\rho(D)$ is defined by

$$\begin{aligned}\omega_\rho(D)\phi(x):&=\left(\mathscr{F}^*\omega_\rho(\xi)\mathscr{F}\phi\right)(x)\\&=\int_{\mathbb{R}^n}e^{ix\cdot\xi}\omega_\rho(\xi)(\mathscr{F}\phi)(\xi)d\xi/(2\pi)^{n/2}\\&=\int_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi}\omega_\rho(\xi)\phi(y)dyd\xi/(2\pi)^n.\end{aligned}$$

*Corresponding author. *Email address:* `aishida@rs.tus.ac.jp` (A. Ishida)

In particular, $\omega_1(D)$ is the usual free Schrödinger operator $-\Delta/2 = -\sum_{j=1}^{n} \partial_{x_j}^2/2$, but we exclude the operator $\omega_{1/2}(D)$ — i.e. the massless relativistic Schrödinger operator $\sqrt{-\Delta}$.

In what follows, the potential function $V = V(x)$ is a real-valued multiplication operator algebraically decaying at the infinity. In this work, we consider the case $V \in L^{\infty}(\mathbb{R})$. A more detailed description of the function V is provided by Condition 2.1. The total Hamiltonian under consideration has the form

$$\omega_\rho(D) + V. \tag{1.1}$$

Due to the boundedness of V, the operator (1.1) considered on the space $L^2(\mathbb{R}^n)$ is self-adjoint. We note that the potential functions of scattering theory usually have a singularity. However, here we would like to present the example of potential function for which standard wave operators do not exist. Therefore, we are not forced to deal with singular functions.

Assume that the free side is represented by the usual free Schrödinger operator

$$|D|^2/2 = -\Delta/2.$$

If

$$|V(x)| \lesssim \langle x \rangle^{-\gamma}, \tag{1.2}$$

where $\gamma > 0$ and $\langle x \rangle := \sqrt{1+|x|^2}$, then the wave operators exist for $\gamma > 1$ and are absent for $\gamma \leqslant 1$ — cf. [4, 12]. Thus $\gamma = 1$ is the borderline between the short- and long-range behaviour. The classical trajectory of the particle in the dynamics of the free Schrödinger equation satisfies the condition $x(t) = \mathscr{O}(t)$ as $t \to \infty$. Moreover, for $\phi \in L^2(\mathbb{R}^n)$ the Cook-Kuroda method states that if

$$\int_1^{\infty} \left\| V e^{-it|D|^2/2} \phi \right\| dt < \infty,$$

where $\|\cdot\|$ is the usual L^2-norm, then the wave operators

$$\operatorname*{s-lim}_{t \to \pm\infty} e^{it(|D|^2/2+V)} e^{-it|D|^2/2}$$

exist. It follows from the estimate

$$\begin{aligned} &\left\| \left(e^{it_1(|D|^2/2+V)} e^{-it_1|D|^2/2} - e^{it_2(|D|^2/2+V)} e^{-it_2|D|^2/2} \right) \phi \right\| \\ &\leqslant \int_{t_2}^{t_1} \left\| \partial_t \left(e^{it(|D|^2/2+V)} e^{-it|D|^2/2} \right) \phi \right\| dt = \int_{t_2}^{t_1} \left\| V e^{-it|D|^2/2} \phi \right\| dt \longrightarrow 0 \end{aligned}$$

as $t_1, t_2 \to \infty$. Taking into account the decay condition (1.2), we can formally check the borderline condition by substituting the functions $x(t) = \mathscr{O}(t)$ into $V(x)$, so that

$$\int_1^{\infty} \left\| V e^{-it|D|^2/2} \phi \right\| dt = \int_1^{\infty} \left\| V(x(t)) e^{-it|D|^2/2} \phi \right\| dt \lesssim \int_1^{\infty} t^{-\gamma} dt. \tag{1.3}$$