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Linearised Estimate of the Backward Error for Equality Constrained Indefinite Least Squares Problems

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Abstract. We study normwise backward error estimates for the equality constrained indefinite least squares problem and introduce an explicit sub-optimal linearisation estimate. Numerical examples show the reliability and efficiency of the estimate presented.

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Key words: Indefinite least squares, equality constrained indefinite least squares problem, normwise backward error, linearisation estimate.

1. Introduction

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $m \geq n$ and I_p be the $p \times p$ identity matrix. We consider the indefinite least squares (ILS) problem

$$\min_x (b - Ax)^\top \Sigma_{pq} (b - Ax), \tag{1.1}$$

where $\top$ refers to the transposition operation and

$$\Sigma_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad p + q = m \tag{1.2}$$

is the signature matrix [2, 3]. The ILSP (1.1) finds various applications in the total least squares problem [20] and optimisation [8, 19]. A generalisation of ILS — viz. the equality constrained indefinite linear least square (ILSE) problem was introduced by Bojanczyk *et al.* [1]. More exactly, if $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $B \in \mathbb{R}^{s\times n}$, $d \in \mathbb{R}^s$, $m \geq n$, and Σ_{pq} is the signature matrix (1.2), then it is the following minimisation problem:

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$$\min_x (b-Ax)^\top \Sigma_{pq}(b-Ax) \quad \text{subject to} \quad Bx=d. \tag{1.3}$$

The conditions of the unique solvability of the ILSE problem (1.3) — viz.

$$\operatorname{rank}(B)=s \quad \text{and} \quad x^\top(A^\top\Sigma_{pq}A)x>0, \tag{1.4}$$

for any x belonging to the null space $\mathcal{N}(B)$ of the matrix B, are established in [1]. The first condition in (1.3) guarantees the solvability of the equality constrain and the second one means that $A^\top\Sigma_{pq}A$ is a positive definite matrix on the space $\mathcal{N}(B)$, so that the solution is unique. If the conditions (1.4) hold, the solution x of (1.3) can be derived from the normal equation

$$A^\top\Sigma_{pq}(b-Ax)=B^\top\xi, \quad Bx=d, \tag{1.5}$$

where ξ is the vector of the Lagrange multipliers. On the other hand, the solution x can be also derived from the augmented system

$$\mathscr{A}\mathbf{x}:=\begin{bmatrix} 0 & 0 & B \\ 0 & \Sigma_{pq} & A \\ B^\top & A^\top & 0 \end{bmatrix}\begin{bmatrix} \lambda \\ s \\ x \end{bmatrix}=\begin{bmatrix} d \\ b \\ 0 \end{bmatrix}=:\mathbf{b}, \tag{1.6}$$

where $s=\Sigma_{pq}r$, $r=b-Ax$ and $\lambda=-\xi$. It was pointed out in [1] that (1.4) yields the invertibility of the matrix $\mathscr{A}$ in (1.6). For numerical algorithms and the theory of ILSE problem, the reader can consult the papers [17, 18].

Backward error analysis plays an important role in the stability of numerical algorithms. Moreover, backward errors can be used in the stopping criteria of iterative methods for large scale problems. The backward error for the linear least squares problem is investigated in [10–12, 21], the scale total least squares (STLS) problem in [4], and the equality constrained least squares (LSE) problem and the least squares problem over a sphere (LSS) in [5, 16]. Since the evaluation of backward errors for least squares problems has high computational cost, linearisation estimates have been proposed [4, 9, 15]. However, to the best of our knowledge, the ILSE normwise backward errors have not yet been studied. Here we introduce such errors in ILSE case and establish the corresponding linearisation estimate.

Condition numbers describe the sensitivity of output data with respect to the perturbations of input data [11]. The problems with large condition numbers are usually called ill-posed [11], which means that approximate solutions are subject to large errors. Backward error is the smallest perturbation of the input data to make the approximate solution an exact solution of the corresponding perturbed problem [11]. Using the backward errors and condition numbers, the forward error can be estimated by the following thumb rule

$$\text{forward error} \lesssim \text{condition number} \times \text{backward error}.$$

This inequality can be sharp — cf. [11, Page 9]. Condition number for ILS (1.1), ILSE (1.3), and linear least squares problem with equality constraints are considered in [6, 7, 13, 14, 22].

The paper is organised as follows. In Section 2, we define the normwise backward error for ILSE and establish its linearisation estimate. To show the efficiency of this estimate, numerical examples are presented in Section 3. Section 4 contains our concluding remarks.