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The Identification of the Time-Dependent Source Term in Time-Fractional Diffusion-Wave Equations

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Abstract. An inverse time-dependent source problem for a multi-dimensional fractional diffusion wave equation is considered. The regularity of the weak solution for the direct problem under strong conditions is studied and the unique solvability of the inverse problem is proved. The regularised variational problem is solved by the conjugate gradient method combined with Morozov's discrepancy principle. Numerical examples show the stability and efficiency of the method.

AMS subject classifications: 65M32, 35R11

Key words: Inverse source problem, fractional diffusion-wave equation, conjugate gradient method.

1. Introduction

Fractional diffusion and fractional diffusion-wave equations often occur in biology, physics, chemistry, biochemistry and so on — cf. Refs. [23, 24, 27, 42, 43]. They are used in the description of diffusion abnormalities such as sub-diffusion and super-diffusion phenomena [3, 24, 29, 32]. For time-fractional diffusion equations various direct and inverse problems are considered in [2,7,13,18,19,22,28,30,33,37,38,44]. On the other hand, for time-fractional diffusion-wave equations, analytic and week solutions are respectively studied in [1, 12] and [28], the unique solvability of semilinear time-fractional wave equations in [16], and approximation methods in [4, 6, 7]. Hendy *et al.* [11] and Pimenov *et al.* [26] extended the Crank-Nicolson scheme to non-linearly distributed orders in time fractional diffusion-wave equation.

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Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. In particular, we assume that $\partial\Omega \in C^2$ if $d \leq 3$. Moreover, let $T > 0$ be a fixed final time, $\Gamma(\cdot)$ the Gamma function, ∂_{0+}^{α}, $\alpha \in (1,2)$ the Caputo fractional left-sided derivative of order α defined by

$$\partial_{0+}^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,\tau)}{\partial \tau^2} \frac{d\tau}{(t-\tau)^{\alpha-1}}, \quad 0 < t \leq T,$$

and $-L$ a symmetric uniformly elliptic operator,

$$Lu(x) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} A_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x), \quad x \in \Omega$$

such that

$$\begin{aligned} &A_{ij}(x) = A_{ji}(x) \in C^{\infty}(\overline{\Omega}), \quad 1 \leq i,j \leq d, \\ &\mu \sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} A_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega}, \quad (\xi_1,\cdots,\xi_d) \in \mathbb{R}^d, \quad \text{for a constant} \quad \mu > 0, \\ &c(x) \leq 0, \quad x \in \overline{\Omega}, \quad c(x) \in C^{\infty}(\overline{\Omega}), \end{aligned}$$

and

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{d} A_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i,$$

where $\nu = (\nu_1,\cdots,\nu_d) \in \mathbb{R}^d$ is the outward normal unit vector to $\partial\Omega$.

We consider the time-fractional diffusion-wave problem

$$\begin{aligned} &\partial_{0+}^{\alpha} u(x,t) - Lu(x,t) = g(x)h(t), && x \in \Omega, \quad t \in (0,T], \\ &u(x,0) = a(x), && x \in \Omega, \\ &u_t(x,0) = b(x), && x \in \Omega, \\ &\frac{\partial u(x,t)}{\partial \nu} = 0, && x \in \partial\Omega, \quad t \in (0,T]. \end{aligned} \tag{1.1}$$

If the functions $g(x)$, $h(t)$, $a(x)$ and $b(x)$ are known, the system (1.1) represents a direct problem. On the other hand, the inverse problem consists in finding the time-dependent source term $h(t)$ in (1.1) from additional data

$$u(x,t) = f(x,t), \quad x \in \Gamma_0, \quad 0 < t \leq T, \tag{1.2}$$

where Γ_0 is a nonempty subset of the boundary $\partial\Omega$.

The inverse source problem for time-fractional diffusion equation is widely studied. Thus Zhang *et al.* [44] found the solution of an inverse space-dependent source problem from the Cauchy data at $x = 0$, Sakamoto *et al.* [28] established a stability estimate for identification of a time-dependent source from the measurements at an interior