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A Fractional Tikhonov Regularisation Method for Finding Source Terms in a Time-Fractional Radial Heat Equation

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Abstract. The ill-posed problems of the identification of unknown source terms in a time-fractional radial heat conduction problem are studied. To overcome the difficulties caused by the ill-posedness, a fractional Tikhonov regularisation method is proposed. Employing Mittag-Leffler function, we obtain error estimates under a priori and a posteriori regularisation parameter choices. In the last situation, the Morozov discrepancy principle is also used. Numerical examples show that the method is a stable and effective tool in the reconstruction of smooth and non-smooth source terms and, generally, outperforms the classical Tikhonov regularisation.

AMS subject classifications: 35R25, 35R30, 65J20, 65M30

Key words: Inverse source problem, fractional Tikhonov regularisation method, error estimate.

1. Introduction

Fractional diffusion equations are widely used in non-Markovian diffusion processes — i.e. in diffusion phenomena with memory [2, 17, 18]. The replacement of the standard time derivatives by fractional ones leads to time-fractional diffusion equations, which can be used to describe superdiffusion and subdiffusion phenomena [24,25]. Initial and boundary value problems for such equations are well studied — cf. Refs. [10, 15, 16, 20].

Nevertheless, in practical applications certain data, such as the source term, diffusion coefficients, portions of initial and boundary conditions can be missing and we try to recover them from additional measurements. This procedure generates an inverse problem in fractional diffusion [11,13,19]. The physical model considered here is given on a radial symmetric plate of radius r_0 under the assumption that the initial temperature is axisymmetric. Mathematically, it is described by the following initial boundary problem for the

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time-fractional radial heat equation

$$D_t^\alpha u(r,t) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + f(r), \quad 0 < r < r_0, \quad 0 < t < T, \tag{1.1}$$

$$u(r,0) = 0, \quad 0 < r \le r_0, \tag{1.2}$$

$$u(r_0,t) = 0, \quad 0 \le t \le T, \tag{1.3}$$

where $D_t^\alpha u(r,t)$ is the Caputo fractional derivative of order α, $0 < \alpha \le 1$ defined by

$$D_t^\alpha u(r,t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{u_\tau(r,\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\ u_t(r,t), & \alpha = 1, \end{cases}$$

and $\lim_{r\to 0} u(r,t)$ is a bounded function on the interval $[0,T]$.

The data appear as the final observation

$$u(r,T) = g(r), \quad 0 \le r \le r_0, \tag{1.4}$$

and the inverse problem consists in the reconstruction of the source term $f(r)$. Note that this is an ill-posed problem — cf. Section 4.

The inverse source problem has been vigorously studied. Thus Wang *et al.* [30] applied reproducing kernel space method to solve an inverse space-dependent source problem, Wei and Wang [31] used a modified quasi-boundary value method, Zhang and Xu [37] employed the Cauchy data at one end, Tatar *et al.* [26, 27] considered it for a space-time fractional diffusion equation and investigated a nonlocal inverse source problem, Cheng *et al.* [4] used a spectral method to determine an unknown heat source term from the final temperature history in the radial domain and provided logarithmic-type error estimates for regularised solutions, Xiong and Ma [33] discussed a backward ill-posed problem for an axis-symmetric fractional diffusion equation. For other relevant results, the reader is referred to [3, 9, 12, 14, 28, 34–36].

The study of inverse source problems for radial heat equation meets a number of difficulties. Thus this equation is connected to the Mittag-Leffler function, so that the error evaluation requires more efforts. On the other hand, this equation is more complex than the classical one. Here we use the regularisation approach [6, 8]. It can be considered as an "interpolation" between classical quasi-boundary method [32] and the Tikhonov regularisation method.

The remainder of this paper is organised as follows. In Section 2, we recall auxiliary results, which are needed for what follows. The solution of the problem (1.1)-(1.4) is presented in Section 3. Section 4 introduces a fractional Tikhonov regularisation method and provides convergence estimates for an a priori and a posteriori regularisation parameter choices. Numerical results are presented in Section 5 and our conclusion is Section 6.