

A Fast Finite Volume Method on Locally Refined Meshes for Fractional Diffusion Equations

Jinhong Jia¹ and Hong Wang^{2,*}

¹*School of Mathematics and Statistics, Shandong Normal University, Jinan 250358, Shandong, China.*

²*Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.*

Received 27 November 2018; Accepted (in revised version) 28 March 2019.

Abstract. In this work, we consider a boundary value problem involving Caputo derivatives defined in the plane. We develop a fast locally refined finite volume method for variable-coefficient conservative space-fractional diffusion equations in the plane to resolve boundary layers of the solutions. Numerical results are presented to show the utility of the method.

AMS subject classifications: 34A08, 65F10, 65M08

Key words: Space-fractional diffusion equation, locally refined mesh, Toeplitz matrix, circulant matrix, finite volume method.

1. Introduction

Fractional partial differential equations (FPDEs), extensively investigated in recent decades, provide powerful and flexible means for modeling anomalously diffusive transport and long-range spatial interaction [26, 28, 29, 42]. Subsequently, the set of numerical methods developed for their solution includes finite difference method (FDM), finite element method (FEM), finite volume method (FVM), and mixed FEM [2, 4, 6, 7, 13, 18, 20–23, 27, 32, 35, 45, 46].

In their pioneer works, Ervin *et al.* [9, 11, 12] proved the well-posedness of the Galerkin weak formulation for space-fractional PDEs with a constant diffusivity. In addition, they established optimal error estimates of the FEM in the energy- and L^2 -norms under the assumptions that the exact solutions to FPDE and adjoint PFDE are smooth enough. On the other hand, as is shown in [19, 39, 41] the solution to the homogeneous Dirichlet boundary-value problem for one-dimensional fractional diffusion equation (FDE) with smooth diffusivity and right-hand side may exhibit boundary layer and have poor regularity. Consequently, there is no known verifiable condition to provide the required smoothness

*Corresponding author. *Email addresses:* jhjia@sdsu.edu.cn (J. Jia), hwang@math.sc.edu (H. Wang)

of the solution and ensure an optimal-order convergence of the Galerkin FEM as required in [9, 11, 12].

Since then, different numerical methods have been developed for solving FPDEs with boundary layers. In particular, in order to handle boundary layers in solutions of homogeneous Dirichlet boundary-value problem for one-dimensional constant-coefficient fractional diffusion equation (FDE), generalised Jacobi polynomials (poly-fractionomials) have been introduced [3, 10, 24, 25, 43, 44]. These polynomials are the eigenfunctions of a singular fractional Sturm-Liouville problem and the resulting spectral methods have diagonal stiffness matrices, which are easily invertible. Most important, an optimal-order error estimate can be established if the right-hand side is smooth in a weighted L^2 -norm. The results are also substantiated by the corresponding numerical experiments.

In addition, for the inhomogeneous Dirichlet boundary value problem for one-sided variable-coefficient conservative FDE, an indirect Galerkin FEM and spectral Galerkin methods are developed [40, 41]. These works exploit an idea from [38] and express the solutions of FDEs as the fractional derivative of the solutions of a second-order diffusion equation. Algorithmically, the second-order diffusion equation is solved by an FEM or a spectral Galerkin method such that the smoothness of diffusivity and the right-hand side ensures the required smoothness of the solution and also the high-order convergence of the numerical methods used. Then a fractional differentiation of the corresponding Galerkin FEM or spectral Galerkin solution is performed to obtain an approximation to the FDE, which would retain the high-order convergence of the postprocessed solution in the L^2 -norm. Numerical experiments show the strong potential of these methods.

A rather natural approach to boundary layers in solutions of FPDEs is to develop numerical approximations on locally refined meshes. However, FPDEs contains complex integral operators with singular kernels, so that the corresponding FDM, FEM, and FVM generate full or dense stiffness matrices and if N denotes the number of spatial freedoms, the traditionally used direct solvers require $\mathcal{O}(N^2)$ memory and $\mathcal{O}(N^3)$ computational complexity [8, 31, 34, 36]. Such computational complexity and memory requirements render realistic multidimensional FPDE modeling and simulations computationally intractable.

It was discovered in [37] that the stiffness matrix for the Meerschaert-Tadjeran FDM for a one-dimensional space-fractional PDE has a Toeplitz-like structure. Consequently, for solving FPDEs fast Krylov subspace iterative methods with preconditioners using Toeplitz-matrix structures have been developed. Although such methods have $\mathcal{O}(N)$ memory requirements and an almost linear computational complexity [8, 31, 33, 34], they deal with discretisation on uniform spatial meshes. For gridded meshes, the corresponding FVM can be expressed as the product of Toeplitz and diagonal matrices, with the latter responsible for the impact of different mesh sizes [16]. It was also shown that for locally refined composite meshes, the stiffness matrix loses its Toeplitz structure due to the nonlocal nature of fractional differential operators [17]. However, the stiffness matrix can be approximated by a block diagonal Toeplitz-like matrix plus a low-rank matrix. Consequently, a fast Krylov subspace iterative method and a circulant matrix based preconditioner were developed. Numerical experiments show that for solutions with boundary layers, fast FVMs perform much better than those using uniform meshes.