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# High Accuracy Analysis of an Anisotropic Nonconforming Finite Element Method for Two-Dimensional Time Fractional Wave Equation

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**Abstract.** High-order numerical analysis of a nonconforming finite element method on regular and anisotropic meshes for two dimensional time fractional wave equation is presented. The stability of a fully-discrete approximate scheme based on quasi-Wilson FEM in spatial direction and Crank-Nicolson approximation in temporal direction is proved and spatial global superconvergence and temporal convergence order $\mathscr{O}(h^2 + \tau^{3-\alpha})$ in the broken $H^1$-norm is established. For regular and anisotropic meshes, numerical examples are consistent with theoretical results.

**AMS subject classifications**: 65N30, 65N15

**Key words**: Time fractional wave equation, anisotropic nonconforming quasi-Wilson finite element, Crank-Nicolson scheme, stability, superclose and superconvergence.

## 1. Introduction

Fractional differential equations have recently attracted increasing attention in various fields of science and engineering. They play an important role in anomalous transport modeling and in theory of complex systems — cf. [4, 6, 15, 22, 28, 29, 34, 35]. However, analytical solutions of such equations are rarely available and even if they are known, their computation meets essential difficulties because of the presence of special functions. This led to the development of various numerical methods. In particular, for time-fractional

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diffusion equations, finite difference methods are considered in [7,19,24,42,44,46], Petrov-Galerkin methods in [31], DG methods in [16] and finite element methods in [21,23,38, 48–50].

Let $\Omega \subset R^2$ be a bounded convex polygonal region with the boundary $\partial\Omega$ and let

$$H^2(H^2(\Omega)) := \left\{\varphi(\mathbf{x},t) : \|\varphi(\mathbf{x},t)\|_2 \in H^2[0,T]\right\}.$$

In this work, we apply a spatial finite element method and temporal Crank-Nicolson scheme to the following two-dimensional time fractional wave equation (TFWE):

$$\begin{aligned}
&D_t^\alpha u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t), && (\mathbf{x},t) \in \Omega \times (0,T],\\
&u(\mathbf{x},t) = 0, && (\mathbf{x},t) \in \partial\Omega \times (0,T],\\
&u(\mathbf{x},0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x},0) = \tilde{u}_0(\mathbf{x}), && \mathbf{x} \in \Omega,
\end{aligned} \tag{1.1}$$

where $\mathbf{x} = (x,y), u_0(\mathbf{x}), \tilde{u}_0(\mathbf{x})$ and $f(\mathbf{x},t)$ are sufficiently smooth functions, $u(\mathbf{x},t) \in H^2(H^2(\Omega))$, the operator $D_t^\alpha$ is the left-sided Caputo fractional derivative of order $\alpha$ with respect to $t$ defined by

$$D_t^\alpha u(\mathbf{x},t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\mathbf{x},s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad 1 < \alpha < 2,$$

and $\Gamma$ is the Gamma function [22].

TFWEs represent a generalisation of classical diffusion and wave equations arising in modeling of diffusion and waves in fluid flow and oil strata. Numerical methods are widely used in their solution. Thus Bhrawy *et al.* [5], developed an efficient and accurate spectral numerical method for second- and fourth-order TFWEs and TFWEs with damping involving the Jacobi tau spectral procedure and Jacobi operational matrix for fractional integrals. Sun and Wu [40] introduced a fully-discrete scheme for TFWEs and proved that it is solvable, unconditionally stable and converges in $L_\infty$-norm. Based on the Crank-Nicolson method combined with the $L1$-approximation, Fairweather *et al.* [12] developed an alternating direction implicit (ADI) orthogonal spline collocation method for two-dimensional TFWEs and established its optimal accuracy in various norms. Du *et al.* [9] considered higher order difference methods for TFWEs, proving their unconditional stability and convergence in $L_\infty$-norm. Zhang *et al.* [47] established unconditional stability and convergence of a compact ADI difference scheme and a Crank-Nicolson ADI scheme for two-dimensional TFWE. Ding and Li [8] approximated Riemann-Liouville derivative by second and fourth order difference schemes and constructed two difference schemes for TFDWEs with reaction term. Huang *et al.* [17] used partial integro-differential equations in finite difference schemes for initial-boundary value TFWEs and proved their convergence with first order accuracy in temporal and second order accuracy in spatial directions. Zeng [44] proposed stable and conditionally stable finite difference schemes for the TFDWE, based on fractional trapezoidal rules and second order generalized Newton-Gregory formulas and central differences. The approximation method developed by Yang *et al.* [43] is based on the transformation of TFWEs into an integro-differential equation and on Lubich fractional