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Multiple Positive Solutions to Singular Fractional Differential System with Riemann-Stieltjes Integral Boundary Condition

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Abstract: In this paper, we study a class of singular fractional differential system with Riemann-Stieltjes integral boundary condition by constructing a new cone and using Leggett-Williams fixed point theorem. The existence of multiple positive solutions is obtained. An example is presented to illustrate our main results.

Key words: fractional differential equation, positive solution, integral boundary condition, fixed point theorem

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1 Introduction

In recent years, the theory of integral boundary value problems have been studied widely and there are many excellent results (see [1]–[11] and references therein). In [12], by means of the fixed point index theory in cones, the authors obtained the existence and multiplicity of positive solutions for a singular fractional differences equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+q(t)f(t,\, u(t))=0, \qquad 0<t<1,\ n-1<\alpha\leq n, \\ u^{(k)}(0)=0, \quad 0\leq k\leq n-2, \qquad u(1)=\displaystyle\int_0^1 u(s)\mathrm{d}A(s), \end{cases} \tag{1.1}$$

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where $\alpha \geq 2$, D_{0+}^{α} is standard Riemann-Liouville derivative, $q(t)$ may be singular at $t = 0$ or $t = 1$, $f(t, u)$ may also be singular at $u = 0$, $\int_0^1 u(s)\mathrm{d}A(s)$ denotes the Riemann-Stieltjes integral, A is a function of bounded variation.

Recently, the coupled boundary value problems to fractional differential systems were caught much attention. For example, Henderson and Luca[13] investigated the nonexistence of a system of nonlinear Riemann-Liouville fractional differential equations with coupled integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t), v(t)) = 0, & 0 < t < 1,\ n-1 < \alpha \leq n, \\ D_{0+}^{\beta} v(t) + \mu g(t, u(t), v(t)) = 0, & 0 < t < 1,\ m-1 < \beta \leq m, \\ u^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, & u(1) = \displaystyle\int_0^1 v(s)\mathrm{d}H(s), \\ v^{(j)}(0) = 0, \quad 0 \leq j \leq m-2, & v(1) = \displaystyle\int_0^1 u(s)\mathrm{d}K(s), \end{cases} \tag{1.2}$$

where D_{0+}^{α} and D_{0+}^{β} denote Riemann-Liouville derivative of orders α and β, $\int_0^1 v(s)\mathrm{d}H(s)$ and $\int_0^1 u(s)\mathrm{d}K(s)$ are the Riemann-Stiegjes integrals.

Henderson *et al.*[14]–[15] also studied the following system with uncoupled integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t), v(t)) = 0, & 0 < t < 1,\ n-1 < \alpha \leq n, \\ D_{0+}^{\beta} v(t) + \mu g(t, u(t), v(t)) = 0, & 0 < t < 1,\ m-1 < \beta \leq m, \\ u^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, & u(1) = \displaystyle\int_0^1 u(s)\mathrm{d}H(s), \\ v^{(j)}(0) = 0, \quad 0 \leq j \leq m-2, & v(1) = \displaystyle\int_0^1 v(s)\mathrm{d}K(s). \end{cases} \tag{1.3}$$

Motivated by the aforementioned papers, a natural question is whether the coupled and the uncoupled integral boundary conditions can be unified in a system. If we have unified the conditions, how can we obtain the existence of the solution? This question motivates the study of the existence of multiple positive solutions for the following singular boundary value problems of fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha_1} u(t) + a_1(t) f_1(t, u(t), v(t)) = 0, & 0 < t < 1,\ n_1 - 1 < \alpha_1 \leq n_1, \\ D_{0+}^{\alpha_2} v(t) + a_2(t) f_2(t, u(t), v(t)) = 0, & 0 < t < 1,\ n_2 - 1 < \alpha_2 \leq n_2, \\ u^{(k_1)}(0) = 0, \quad 0 \leq k_1 \leq n_1 - 2, & u(1) = g_1(\beta_1[u], \beta_1[v]), \\ v^{(k_2)}(0) = 0, \quad 0 \leq k_2 \leq n_2 - 2, & v(1) = g_2(\beta_2[u], \beta_2[v]), \end{cases} \tag{1.4}$$

where $\alpha_i > 2$, $D_{0+}^{\alpha_i}$ is the standard Riemann-Liouville derivative, $a_i(t) \in C((0, 1), [0, +\infty))$ may be singular at $t = 0$ and/or $t = 1$, $f_i \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$, $n_i \in \mathbf{Z}$, $n_i \geq 3$, $i = 1, 2$. The functionals $g_i \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$ are linear functionals