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Using Parity to Accelerate the Computation of the Zeros of Truncated Legendre and Gegenbauer Polynomial Series and Gaussian Quadrature

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Abstract. Any function $u(x)$ can be decomposed into its parts that are symmetric and antisymmetric with respect to the origin. The zeros, maxima and minima of a truncated spectral series of degree N can always be computed as the eigenvalues of the sparse N-dimensional companion matrix whose elements are trivial functions of the coefficients of the spectral series. Here, we show that the matrix dimension can be halved if the series has definite parity. A series of Legendre and Gegenbauer polynomials has even parity if only even degree coefficients are nonzero and odd parity if the sum includes odd degrees only. We give the elements of the parity-exploiting companion matrices explicitly. We also give the coefficients of parity-exploiting recurrences for computing the orthogonal polynomials of even degree only or odd degree only without the wasteful computation of all polynomials of the opposite parity. For an N-point Gaussian quadrature, the quadrature points are the eigenvalues of a symmetric tridiagonal matrix of dimension N ("Jacobi matrix"). We give the explicit elements of symmetric tridiagonal matrices of dimension $N/2$ that do the same job.

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1. Introduction

Gaussian quadrature is a method for numerical approximation of integrals [10, 20]. After the weights, abscissas and samples of the integrand at the abscissas have been computed, the final step in classical Gaussian quadrature is

$$\int_{-1}^{1} u(x)dx \approx \sum_{n=1}^{N} w_n u(x_n) \quad \text{Classical.} \tag{1.1}$$

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This final step (1.1) costs N additions and N multiplications. The parity-exploiting formulas are (for N even)

$$\int_{-1}^{1} u(x)dx \approx \sum_{n=1}^{N/2} w_{N/2+n} \left\{ u(x_{N/2+n}) + u(x_{N/2-n+1}) \right\} \quad \text{Parity}[N \text{ even}], \tag{1.2}$$

whereas for odd N, $x = 0$ is always an abscissa and the fast formula is, with $M \equiv (N-1)/2$,

$$\int_{-1}^{1} u(x)dx \approx w_{M+1}u(0) + \sum_{n=1}^{M} w_{n+1+M} \left\{ u(x_{n+1+M}) + u(x_{M+1-n}) \right\} \quad [N \text{ odd}].$$

This more efficient formula (1.2) requires the same number of additions but only half as many multiplications.

Parity is equally fruitful in computing Gaussian quadrature weights and abscissas. The next section provides both formal and informal definitions of parity and catalogues some useful properties.

A robust and simple method for computing the roots of a truncated series of orthogonal polynomials is to form the companion matrix from the coefficients of the spectral series. The desired zeros are just the eigenvalues of the companion matrix. The stationary points of a function (maxima, minima and saddle points) are located at the zeros of the first derivative of the function. Two-term recursions allow one to easily calculate the coefficients of the first derivative from those of the function itself. One can then apply the companion matrix of the derivative to compute the location of all stationary points.

In this article, the focus is the exploitation of parity for root-finding and Gaussian quadrature for Gegenbauer and Legendre polynomials. The Gegenbauer orthogonal polynomials of order $\mathfrak{m}$ are the set of polynomials orthogonal on the interval $x \in [-1, 1]$ with the weight function

$$\omega(x; \mathfrak{m}) \equiv \left(1 - x^2\right)^{\mathfrak{m}-1/2}.$$

Legendre polynomials are the very important special case $\mathfrak{m} = 1/2$, which implies that $\omega(x; \mathfrak{m} = 1/2) \equiv 1$.

Chebyshev polynomials are the special case $\mathfrak{m} = 0$. We need not discuss them here because the Chebyshev quadrature points and weights are known analytically in explicit form [4] and the root-finding problem has been analysed in depth in the author's book [6].

In Section 4, we review the construction of the companion matrices for general orthogonal polynomials. All sets of orthogonal polynomials have a set-specific three-term recursion which is the sole ingredient required, in addition to the coefficients of the truncated spectral series, to compute the corresponding companion matrix.

The computation of Gaussian quadrature abscissas (sampling points) for an N-point quadrature is a special case of the general root-finding problem: the N-th spectral coefficient $a_N = 1$ and all other spectral coefficients are zero. For Legendre and Gegenbauer (ultraspherical) polynomials, parity can halve the dimension of this Gaussian quadrature companion matrix. Section 3 reviews some useful properties of companion matrices and